ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2023, Vol. 17, No. 4, pp. 828–841. © Pleiades Publishing, Ltd., 2023.
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Bäcklund Transformations of the Relativistic Schrödinger Equation

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Received July 23, 2023; revised October 12, 2023; accepted November 1, 2023

Abstract—We study the system of equations obtained on the basis of the relativistic Schrödinger equation and relating the potential, amplitude, and phase functions. Using the methods of the theory of consistency of systems of partial differential equations, we obtain completely integrable systems that relate only two functions of the above three. The systems found are related by Bäcklund transformations.

Keywords: *relativistic Schrödinger equation, Bäcklund transformation, consistency condition*

DOI: 10.1134/S1990478923040129

INTRODUCTION

Methods of group analysis of differential equations [1–4], the method of differential substitutions [5], and, more generally, the method of Lie–Bäcklund transformations [6, 7] in the jet space of the extended equation [8, 9] are used to construct exact solutions of differential equations, study their symmetry properties and search for conservation laws. In practice, as a rule, there are more particular cases of Lie–Bäcklund transformations—differential correspondences between two systems of differential equations, called Bäcklund transformations [10–12]. Such correspondences represent a differential connection between two systems of differential equations that allows one, using a known solution of one system, to constructively find a solution of the second system. Bäcklund transformations for certain equations usually have a name: Laplace cascade method, Euler–Darboux transformation, Bianchi transformation, Mutard transformation, Hirota bilinear method, etc. [2–4, 13, 14]. For example, the Hopf–Cole transformation connects the heat equation and the Burgers equation [15]. The Miura transformation connects mKdV and KdV [16]. Note that, as a rule, the direct and inverse Bäcklund transformations have different qualitative properties. For example, the Hopf–Cole differential substitution $u = 2v_x/v$ transformations the heat equation $v_t = v_{xx}$ into the Burgers equations $u_t = uu_x + u_{xx}$, and the reverse transition is associated with nonlocal resolution $v = \exp(-\frac{1}{2}\int u dx)$. The sweep method is based on the Hopf–Cole transformation [17]. Using this replacement, general solutions of differential equations and systems of second-order differential equations were found for horizontally layered media. Bäcklund transformations are used in constructing soliton solutions of nonlinear equations, studying symmetries and conservation laws, and are also the most important tool in the study of partial differential equations and represent mappings connecting different solutions of these equations [13]. Early studies of such mappings for the nonlinear Schrödinger equation can be found in [22–24]. The interest in Bäcklund transformations is also due to the discovered connections with quantum integrable systems and the phenomenon of separation of variables [25, 26]. Finding Bäcklund correspondences for actual equations of mathematical physics is a labor-intensive independent task. Examples of Bäcklund transformations and their applications can be found in [13, 14, 27].

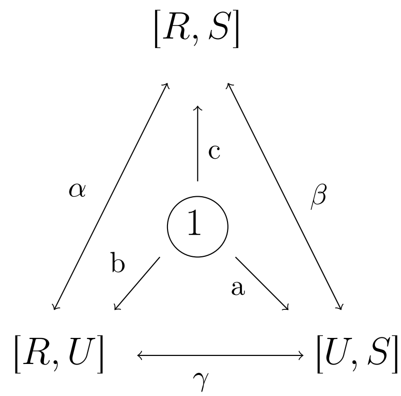

Fig. 1. Eliminating one of the functions U, R, and S from relation (1).

1. STATEMENT OF THE PROBLEM

Let us consider the Klein–Gordon–Fock equation [28] (relativistic Schrödinger equation)

$$W_{tt} = W_{xx} + UW, \tag{1}$$

where $W = Re^{iS}$, i is the imaginary unit, and U, R, and S are some real-valued functions of the variables t and x. For brevity, we use the notation $W_t = \frac{\partial W}{\partial t}$, $W_{tt} = \frac{\partial^2 W}{\partial t^2}$, $W_x = \frac{\partial W}{\partial x}$, $W_{xx} = \frac{\partial^2 W}{\partial x^2}$, etc. for partial derivatives. The functions U, R, and S are called potential, amplitude, and phase, respectively. We consider the problem of eliminating one of the functions U, R, and S from relation (1). Such transformations are part of the general theory of overdetermined systems of partial differential equations. Conventionally, such transformations can be expressed by the diagram in Fig. 1, where $[R, U]$, $[U, S]$, and $[R, S]$ denote systems of differential equations obtained by eliminating the functions S, R, and U from the Eq. (1), arrows a, b, and c correspond to the exclusion of functions R, S, and U from Eq. (1), and arrows α, β, and γ, essentially, are Bäcklund transformations.

Note that a similar problem was considered in [29] for the classical Schrödinger equation.

Equation (1) for a complex-valued function W is equivalent to two real equations in the form of the system

$$\begin{aligned} R_{tt} - RS_t^2 &= R_{xx} - RS_x^2 + UR, \\ 2R_tS_t + RS_{tt} &= 2R_xS_x + RS_{xx}. \end{aligned} \tag{2}$$

In this paper, we derive the differential relations $[U, S]$ (Theorem 1), containing only the functions U and S, and the relations $[U, R]$ (Theorem 2), containing only the functions U and R. In this case, we use the consistency theory algorithm of bringing the overdetermined system into involution. The transition from the relation $[U, S]$ to the relation $[U, R]$ is carried out by introducing differential relations for the function R and, in fact, represents a Bäcklund transformation [10]. The inverse transition from the relation $[U, R]$ to the relation $[U, S]$ is carried out by introducing differential relations for the function S and represents the inverse Bäcklund transformation.

If we assume that the functions U and R in (2) are known, then this will be an overdetermined system for the function S. If U and S are known, then this will be an overdetermined system for the function R. In each of these cases, system (2) is solvable under additional conditions—consistency conditions for the known functions. The main goal of the present paper work is to find these consistency conditions. As a corollary, we obtain completely integrable systems the solution of which yields an exact solution of Eq. (1).

Examples of constructing exact solutions for the system $[R, S]$ are given. Reconstructing the function U from the amplitude R and phase S amounts to solving the inverse problem, where, given a scattered particle/wave with parameters (R, S) one can reconstruct the shape of the potential well U.

All functions considered are assumed to be analytic.

2. TRANSFORMATION a—ELIMINATING THE FUNCTION R FROM SYSTEM (2)

Let us rewrite (2) in the form

$$R_{tt} = R_{xx} + R(S_t^2 - S_x^2 + U), \quad R_t = R_x \frac{S_x}{S_t} + R\frac{S_{xx} - S_{tt}}{2S_t}. \tag{3}$$

The following theorem holds.

Theorem 1. *Let functions A, B, C, D_0, D_1, E_0, E_1, F_0, F_1, G_0, G_1, H_0, and H_1 be defined based on the functions $S = S(t,x)$ and $U = U(t,x)$ by the following recurrence system of relations*:

$$A = S_t^2 - S_x^2 + U, \quad B = \frac{S_x}{S_t}, \quad C = \frac{S_{xx} - S_{tt}}{2S_t}, \tag{4}$$

$$D_1 = \frac{B_t + 2BC + BB_x}{1 - B^2}, \quad D_0 = \frac{BC_x + C_t + C^2 - A}{1 - B^2}, \tag{5}$$

$$E_1 = B_x + BD_1 + C, \quad E_0 = C_x + BD_0, \tag{6}$$

$$F_1 = D_{1t} + D_1E_1 + D_0B, \quad F_0 = D_{0t} + D_1E_0 + D_0C, \tag{7}$$

$$G_1 = E_{1x} + E_1D_1 + E_0, \quad G_0 = E_{0x} + E_1D_0, \tag{8}$$

$$H_0 = \frac{G_0 - F_0}{F_1 - G_1}, \quad H_1 = BH_0 + C. \tag{9}$$

Then the functions (S,U,R), where

1. *The functions (S,U) are a solution of the equation*

$$H_{1x} = H_{0t},$$

2. *The function R is a solution of the system*

$$R_x = H_0R, \quad R_t = H_1R,$$

which is consistent by virtue of item 1,

are a solution of system (2), *and consequently*, $W = Re^{iS}$ *is a solution of the Klein–Gordon equation* (1) *with the potential U.*

Proof. By virtue of the notation in (4), system (3) takes the form

$$R_{tt} = R_{xx} + AR, \quad R_t = BR_x + CR. \tag{10}$$

From the second relation in (10) we find

$$R_{tx} = B_xR_x + BR_{xx} + C_xR + CR_x = BR_{xx} + (B_x + C)R_x + C_xR, \tag{11}$$

$$R_{tt} = B_t R_x + B R_{tx} + C_t R + C R_t$$

by virtue of (10)

$$= B_t R_x + B R_{tx} + C_t R + C B R_x + C^2 R$$
$$= B R_{tx} + (B_t + BC) R_x + (C_t + C^2) R$$

by virtue of (11)

$$= B^2 R_{xx} + B(B_x + C) R_x + B C_x R + (B_t + BC) R_x + (C_t + C^2) R$$
$$= B^2 R_{xx} + (B_t + 2BC + B B_x) R_x + (B C_x + C_t + C^2) R. \tag{12}$$

From (10) and (12) we obtain

$$R_{xx} + AR = B^2 R_{xx} + (B_t + 2BC + B B_x) R_x + (B C_x + C_t + C^2) R,$$
$$R_{xx} = \frac{B_t + 2BC + B B_x}{1 - B^2} R_x + \frac{B C_x + C_t + C^2 - A}{1 - B^2} R.$$

By virtue of the notation in (5),

$$R_{xx} = D_1 R_x + D_0 R. \tag{13}$$

By virtue of (13), relation (11) takes the form

$$R_{tx} = B D_1 R_x + B D_0 R + (B_x + C) R_x + C_x R = (B D_1 + B_x + C) R_x + (B D_0 + C_x) R.$$

By virtue of the notation in (6),

$$R_{tx} = E_1 R_x + E_0 R. \tag{14}$$

From (13) we find

$$R_{xxt} = D_{1t} R_x + D_1 R_{tx} + D_{0t} R + D_0 R_t.$$

By virtue of (10) and (14),

$$R_{xxt} = D_{1t} R_x + D_1 (E_1 R_x + E_0 R) + D_{0t} R + D_0 (B R_x + C R)$$
$$= (D_{1t} + D_1 E_1 + D_0 B) R_x + (D_{0t} + D_1 E_0 + D_0 C) R.$$

By virtue of the notation in (7),

$$R_{xxt} = F_1 R_x + F_0 R. \tag{15}$$

From (14) we find

$$R_{txx} = E_{1x} R_x + E_1 R_{xx} + E_{0x} R + E_0 R_x.$$

By virtue of (13),

$$R_{txx} = E_{1x} R_x + E_1 (D_1 R_x + D_0 R) + E_{0x} R + E_0 R_x = (E_{1x} + E_1 D_1 + E_0) R_x + (E_{0x} + E_1 D_0) R.$$

By virtue of the notation in (8),

$$R_{txx} = G_1 R_x + G_0 R. \tag{16}$$

From (15) and (16) we find

$$F_1 R_x + F_0 R = G_1 R_x + G_0 R, \quad R_x = \frac{G_0 - F_0}{F_1 - G_1} R.$$

By virtue of the notation in (9),

$$R_x = H_0 R. \tag{17}$$

By virtue of (17), the second relation in (10) takes the form

$$R_t = (B H_0 + C) R.$$

By virtue of the notation in (9),

$$R_t = H_1 R. \tag{18}$$

From (17) and (18), by virtue of the consistency condition $R_{tx} = R_{xt}$, we obtain

$$H_{1x}R + H_1 R_x = H_{0t}R + H_0 R_t.$$

By virtue of (17) and (18),

$$H_{1x}R + H_1 H_0 R = H_{0t}R + H_0 H_1 R;$$

that is,

$$H_{1x} = H_{0t}. \tag{19}$$

Relation (19) is the desired relation for the functions S and U. When it holds true, the function R is found from the joint system (17), (18).

The proof of the theorem is complete. □

3. TRANSFORMATION b—ELIMINATING THE FUNCTION S FROM SYSTEM (2)

The following assertion holds.

Theorem 2. *Let the functions A, B_1, B_2, D_1, D_2, D_3, D_4, E_1, E_2, E_3, E_4, F_1, F_2, F_3, F_5, F_6, F_7, F_8, G_1, G_2, G_3, G_4, H_1, H_3, H_5, H_6, H_7, H_8, ψ, and φ be defined based on the functions $R = R(t,x)$ and $U = U(t,x)$ by the following recurrence system of relations*:

$$A = \frac{R_{tt} - R_{xx}}{R} - U, \quad B_1 = 2\frac{R_x}{R}, \quad B_2 = -2\frac{R_t}{R}, \tag{20}$$

$$D_1 = \frac{A_t}{2A}, \quad D_2 = -\frac{B_2}{A}, \quad D_3 = \frac{A_x}{2A}, \quad D_4 = -\frac{B_1}{A}, \tag{21}$$

$$\begin{aligned} E_1 &= D_3 - AD_4, \\ E_2 &= D_4, \\ E_3 &= D_1, \\ E_4 &= D_2, \end{aligned} \tag{22}$$

$$\begin{aligned} F_1 &= E_{1x} + E_3 D_1 + E_1^2 - 2AE_3E_4, \\ F_2 &= E_3 D_2, \\ F_3 &= E_{2x} + E_4 D_1 + 4E_1E_2 - 2E_3E_4 - 2AE_4^2, \\ F_5 &= E_4 D_2 + 2E_4^2, \\ F_6 &= E_{3x} + E_3 D_3 + E_1 E_3, \\ F_7 &= E_{4x} + E_3 D_4 + E_4 D_3 + 3E_1E_4 + 3E_2E_3, \\ F_8 &= E_4 D_4 + 5E_2E_4, \end{aligned} \tag{23}$$

$$\begin{aligned} G_1 &= E_3 - AE_4, \\ G_2 &= E_4, \\ G_3 &= E_1, \\ G_4 &= E_2, \end{aligned} \tag{24}$$

$$\begin{aligned} H_1 &= D_{1t} + D_3E_1 + D_1G_1 - 2AD_4G_3, \\ H_3 &= D_{2t} + D_3E_2 + D_4E_1 + D_1G_2 + 3D_2G_1 - 2AD_4G_4 + 2D_4G_3, \\ H_5 &= D_4E_2 + 3D_2G_2 + 2D_4G_4, \\ H_6 &= D_{3t} + D_3E_3 + D_1G_3, \\ H_7 &= D_{4t} + D_3E_4 + D_4E_3 + D_1G_4 + 3D_2G_3 + 2D_4G_1, \\ H_8 &= D_4E_4 + 3D_2G_4 + 2D_4G_2, \end{aligned} \tag{25}$$

$$\psi = \sqrt{w^2 - A},$$
$$\varphi = \sqrt{A + \psi^2},$$

where w is a solution of the tenth degree algebraic equation

$$\big((H_1-F_1)w-F_2w^2+(H_3-F_3)w^3+(H_5-F_5)w^5\big)^2 = (w^2-A)\big((H_6-F_6)+(H_7-F_7)w^2+(H_8-F_8)w^4\big)^2.$$

Then the functions (R, U, S), where

1. *The functions (R, U) are a solution of the equation*

$$\varphi_t = \psi_x,$$

2. *The function S is a solution of the system*

$$S_t = \varphi,$$
$$S_x = \psi,$$

consistent by virtue of item 1,

are a solution of system (11), and consequently, $W = Re^{iS}$ is a solution of the Klein–Gordon equation (1) with the potential U.

Proof. We eliminate the function S from (2),

$$S_t^2 = \frac{R_{tt} - R_{xx}}{R} - U + S_x^2, \quad S_{tt} = S_{xx} + 2\frac{R_x}{R}S_x - 2\frac{R_t}{R}S_t. \tag{26}$$

By virtue of the notation in (20), system (26) takes the form

$$S_t = \sqrt{A + S_x^2}, \quad S_{tt} = S_{xx} + B_1 S_x + B_2\sqrt{A + S_x^2}. \tag{27}$$

From the first relation in system (27) we find

$$S_{tx} = \frac{A_x + 2S_xS_{xx}}{2\sqrt{A + S_x^2}}, \tag{28}$$

$$S_{tt} = \frac{A_t + 2S_xS_{tx}}{2\sqrt{A + S_x^2}}$$

by virtue of (28)

$$= \frac{A_t}{2\sqrt{A + S_x^2}} + \frac{S_x}{\sqrt{A + S_x^2}}\frac{A_x + 2S_xS_{xx}}{2\sqrt{A + S_x^2}}$$
$$= \frac{A_t}{2\sqrt{A + S_x^2}} + \frac{S_xA_x}{2(A + S_x^2)} + \frac{S_x^2S_{xx}}{A + S_x^2}. \tag{29}$$

By virtue of (29) and the second relation in (27), we obtain

$$\frac{A_t}{2\sqrt{A + S_x^2}} + \frac{S_xA_x}{2(A + S_x^2)} + \frac{S_x^2S_{xx}}{A + S_x^2} = S_{xx} + B_1S_x + B_2\sqrt{A + S_x^2},$$
$$\frac{AS_{xx}}{A + S_x^2} = \frac{A_t}{2\sqrt{A + S_x^2}} + \frac{S_xA_x}{2(A + S_x^2)} - B_1S_x - B_2\sqrt{A + S_x^2},$$
$$S_{xx} = \frac{A_t}{2A}\sqrt{A + S_x^2} + \frac{A_x}{2A}S_x - \frac{B_1}{A}(A + S_x^2)S_x - \frac{B_2}{A}(A + S_x^2)^{3/2}.$$

Further, we will write all relations in the form

$$f(w) + S_x g(w),$$

where $w = \sqrt{A + S_x^2}$ and f and g are rational functions of w. By virtue of this convention,

$$S_{xx} = \frac{A_t}{2A} w - \frac{B_2}{A} w^3 + S_x \left(\frac{A_x}{2A} - \frac{B_1}{A} w^2 \right).$$

By virtue of the notation in (21), system (27) has been reduced to the form

$$\begin{aligned} S_t &= w, \\ S_{xx} &= D_1 w + D_2 w^3 + S_x \left(D_3 + D_4 w^2 \right). \end{aligned} \tag{30}$$

Let us compose the consistency condition $(S_t)_{xx} = (S_{xx})_t$.

By virtue of system (30), relation (28) takes the form

$$\begin{aligned} S_{tx} &= \frac{A_x}{2} w^{-1} + S_x w^{-1} S_{xx} \\ &= \frac{A_x}{2} w^{-1} + S_x \left(D_1 + D_2 w^2 + S_x (D_3 w^{-1} + D_4 w) \right) \\ &= \frac{A_x}{2} w^{-1} + S_x (D_1 + D_2 w^2) + S_x^2 (D_3 w^{-1} + D_4 w) \end{aligned}$$

(by virtue of relation $S_x^2 = w^2 - A$)

$$\begin{aligned} &= \frac{A_x}{2} w^{-1} + S_x (D_1 + D_2 w^2) + (w^2 - A)(D_3 w^{-1} + D_4 w) \\ &= \frac{A_x}{2} w^{-1} + S_x (D_1 + D_2 w^2) + D_3 w + D_4 w^3 - A D_3 w^{-1} - A D_4 w \\ &= \left(\frac{A_x}{2} - A D_3 \right) w^{-1} + (D_3 - A D_4) w + D_4 w^3 + S_x (D_1 + D_2 w^2). \end{aligned}$$

By virtue of the notation in (21),

$$\frac{A_x}{2} - A D_3 = \frac{A_x}{2} - A \frac{A_x}{2A} = 0.$$

By virtue of the notation in (22),

$$S_{tx} = E_1 w + E_2 w^3 + S_x (E_3 + E_4 w^2). \tag{31}$$

Since $S_t = w$, we have

$$w_x = E_1 w + E_2 w^3 + S_x (E_3 + E_4 w^2).$$

From (31) we find

$$\begin{aligned} S_{txx} &= E_{1x} w + E_{2x} w^3 + S_x (E_{3x} + E_{4x} w^2) + S_{xx} (E_3 + E_4 w^2) + (E_1 w + 3 E_2 w^2 + 2 E_4 S_x w) w_x \\ &= E_{1x} w + E_{2x} w^3 + S_x (E_{3x} + E_{4x} w^2) + (E_3 + E_4 w^2) \big(D_1 w + D_2 w^3 + S_x (D_3 + D_4 w^2) \big) \\ &\quad + (E_1 w + 3 E_2 w^2 + 2 E_4 S_x w) \big(E_1 w + E_2 w^3 + S_x (E_3 + E_4 w^2) \big) \\ &= E_{1x} w + E_{2x} w^3 + S_x (E_{3x} + E_{4x} w^2) + E_3 D_1 w + E_3 D_2 w^2 + E_4 D_1 w^3 + E_4 D_2 w^5 \\ &\quad + S_x (E_3 D_3 + E_3 D_4 w^2 + E_4 D_3 w^2 + E_4 D_4 w^4) + E_1^2 w + E_2 E_1 w^3 + 2 E_1 E_4 w^2 S_x \\ &\quad + E_1 E_2 w^3 + 3 E_2^2 w^5 + 2 E_2 E_4 S_x w^4 \end{aligned}$$

$$
\begin{aligned}
&+ S_x(E_1E_3 + E_1E_4w^2 + 3E_2E_3w^2 + 3E_2E_4w^4) + S_x^2(2E_3E_4w + 2E_4^2w^3)\\
&= (E_{1x} + E_3D_1 + E_1^2)w + E_3D_2w^2 + (E_{2x} + E_4D_1 + 4E_1E_2)w^3 + E_4D_2w^5\\
&\quad + S_x\Big[E_{3x} + E_{4x}w^2 + E_3D_3 + E_3D_4w^2 + E_4D_3w^2 + E_4D_4w^4 + 2E_1E_4w^2 + 2E_2E_4w^4\\
&\qquad + E_1E_3 + E_1E_4w^2 + 3E_2E_3w^2 + 3E_2E_4w^4\Big] + S_x^2(2E_3E_4w + 2E_4^2w^3)\\
&= (E_{1x} + E_3D_1 + E_1^2)w + E_3D_2w^2 + (E_{2x} + E_4D_1 + 4E_1E_2)w^3 + E_4D_2w^5\\
&\quad + S_x\Big[(E_{3x} + E_3D_3 + E_1E_3) + w^2(E_{4x} + E_3D_4 + E_4D_3 + 3E_1E_4 + 3E_2E_3)\\
&\qquad + w^4(E_4D_4 + 5E_2E_4)\Big] + (w^2 - A)(2E_3E_4w + 2E_4^2w^3)\\
&= (E_{1x} + E_3D_1 + E_1^2 - 2AE_3E_4)w + E_3D_2w^2\\
&\quad + (E_{2x} + E_4D_1 + 4E_1E_2 - 2E_3E_4 - 2AE_4^2)w^3 + (E_4D_2 + 2E_4^2)w^5\\
&\quad + S_x\Big[(E_{3x} + E_3D_3 + E_1E_3) + w^2(E_{4x} + E_3D_4 + E_4D_3 + 3E_1E_4 + 3E_2E_3)\\
&\qquad + w^4(E_4D_4 + 5E_2E_4)\Big].
\end{aligned}
$$

By virtue of the notation in (23),

$$
S_{txx} = F_1w + F_2w^2 + F_3w^3 + F_5w^5 + S_x(F_6 + F_7w^2 + F_8w^4). \tag{32}
$$

Further, we find

$$
w_t = \frac{A_t + 2S_xS_{tx}}{2w}
$$

by virtue of (31),

$$
\begin{aligned}
&= \frac{A_t}{2w} + \frac{S_x}{w}\big(E_1w + E_2w^3 + S_x(E_3 + E_4w^2)\big)\\
&= \frac{A_t}{2w} + S_x(E_1 + E_2w^2) + \frac{S_x^2}{w}(E_3 + E_4w^2)\\
&= \frac{A_t}{2w} + S_x(E_1 + E_2w^2) + \frac{w^2 - A}{w}(E_3 + E_4w^2)\\
&= \frac{A_t}{2w} + S_x(E_1 + E_2w^2) + E_3w + E_4w^3 - \frac{AE_3}{w} - AE_4w\\
&= \left(\frac{A_t}{2} - AE_3\right)w^{-1} + (E_3 - AE_4)w + E_4w^3 + S_x(E_1 + E_2w^2)
\end{aligned}
$$

by virtue of (21) and (22),

$$
= (E_3 - AE_4)w + E_4w^3 + S_x(E_1 + E_2w^2).
$$

By virtue of the notation in (24),

$$
w_t = G_1w + G_2w^3 + S_x(G_3 + G_4w^2).
$$

Further, from the second relation in (30), we find $(S_{xx})_t$,

$$
\begin{aligned}
S_{xxt} &= D_{1t}w + D_{2t}w^3 + S_x(D_{3t} + D_{4t}w^2) + S_{tx}(D_{3t} + D_{4t}w^2) + (D_1 + 3D_2w^2 + 2S_xD_4w)w_t\\
&= D_{1t}w + D_{2t}w^3 + S_x(D_{3t} + D_{4t}w^2) + (D_3 + D_4w^2)\big(E_1w + E_2w^3 + S_x(E_3 + E_4w^2)\big)\\
&\quad + (D_1 + 3D_2w^2 + 2S_xD_4w)\big(G_1w + G_2w^3 + S_x(G_3 + G_4w^2)\big)
\end{aligned}
$$

$$
\begin{aligned}
&= (D_{1t}w + D_{2t}w^3) + (D_3 + D_4w^2)(E_1w + E_2w^3) + (D_1 + 3D_2w^2)(G_1w + G_2w^3) \\
&\quad + 2S_x^2 D_4 w(G_3 + G_4w^2) + S_x\Big[(D_{3t} + D_{4t}w^2) + (D_3 + D_4w^2)(E_3 + E_4w^2) \\
&\qquad + (D_1 + 3D_2w^2)(G_3 + G_4w^2) + 2D_4w(G_1w + G_2w^3)\Big] \\
&= D_{1t}w + D_{2t}w^3 + D_3E_1w + D_3E_2w^3 + D_4E_1w^3 + D_4E_2w^5 \\
&\quad + D_1G_1w + D_1G_2w^3 + 3D_2G_1w^3 + 3D_2G_2w^5 + 2D_4(w^2 - A)(G_3 + G_4w^2)w \\
&\quad + S_x\Big[D_{3t} + D_{4t}w^2 + D_3E_3 + D_3E_4w^2 + D_4E_4w^4) \\
&\qquad + D_1G_3 + D_1G_4w^2 + 3D_2G_3w^2 + 3D_2G_4w^4 + 2D_4G_1w^2 + 2D_4G_2w^4\Big] \\
&= w(D_{1t} + D_3E_1 + D_1G_1 - 2AD_4G_3) \\
&\quad + w^3(D_{2t} + D_3E_2 + D_4E_1 + D_1G_2 + 3D_2G_1 - 2AD_4G_4 + 2D_4G_3) \\
&\quad + w^5(D_4E_2 + 3D_2G_2 + 2D_4G_4) + S_x\Big[(D_{3t} + D_3E_3 + D_1G_3) \\
&\qquad + w^2(D_{4t} + D_3E_4 + D_4E_3 + D_1G_4 + 3D_2G_3 + 2D_4G_1) \\
&\qquad\quad + w^4(D_4E_4 + 3D_2G_4 + 2D_4G_2)\Big].
\end{aligned}
$$

By virtue of the notation in (25),

$$S_{xxt} = H_1w + H_3w^3 + H_5w^5 + S_x(H_6 + H_7w^2 + H_8w^4). \tag{33}$$

By virtue of (32) and (33), the consistency condition $S_{txx} = S_{xxt}$ takes the form

$$(H_1-F_1)w-F_2w^2+(H_3-F_3)w^3+(H_5-F_5)w^5+S_x\big((H_6-F_6)+(H_7-F_7)w^2+(H_8-F_8)w^4\big) = 0. \tag{34}$$

From relation (34) we find

$$S_x = \psi,$$

where ψ depends only on the functions R and U and their derivatives. Further, from the first relation in (27) we obtain

$$S_t = \varphi,$$

where

$$\varphi = \sqrt{A + \psi^2},$$

and it remains to compose the consistency condition

$$\varphi_x = \psi_t,$$

which is the desired relation upon the functions R and U.

Note that in order to find S_x from (34), it is necessary to compose the relation

$$(H_1 - F_1)w - F_2w^2 + (H_3 - F_3)w^3 + (H_5 - F_5)w^5 = -S_x\big((H_6 - F_6) + (H_7 - F_7)w^2 + (H_8 - F_8)w^4\big),$$

square it,

$$\big((H_1 - F_1)w - F_2w^2 + (H_3 - F_3)w^3 + (H_5 - F_5)w^5\big)^2 = S_x^2\big((H_6 - F_6) + (H_7 - F_7)w^2 + (H_8 - F_8)w^4\big)^2,$$

substitute $S_x^2 = w^2 - A$,

$$\big((H_1-F_1)w-F_2w^2+(H_3-F_3)w^3+(H_5-F_5)w^5\big)^2 = (w^2-A)\big((H_6-F_6)+(H_7-F_7)w^2+(H_8-F_8)w^4\big)^2$$

solve the resulting equation of degree 10 for w, and find

$$S_x = \sqrt{w^2 - A},$$

a function of R and U and their derivatives.

The proof of the theorem is complete. □

4. TRANSFORMATION c—ELIMINATING THE FUNCTION U FROM SYSTEM (2)

From the first relation in (2), we find

$$U = \frac{1}{R}(R_{tt} - R_{xx}) + (S_x^2 - S_t^2). \tag{35}$$

For the functions R and S there remains the second relation in system (2).

$$S_{tt} - S_{xx} = 2\left(\frac{R_x}{R}S_x - \frac{R_t}{R}S_t\right). \tag{36}$$

Example 1. If R and S are a pair of harmonically conjugate functions

$$\begin{aligned} R_t &= S_x, \\ R_x &= -S_t, \end{aligned}$$

then (36) takes the form

$$S_{tt} - S_{xx} = 0.$$

Consequently,

$$S = f(x+t) + g(x-t),$$

where f and g is some function of a single argument. Then for the function R we have the system

$$\begin{aligned} R_t &= f'(x+t) + g'(x-t), \\ R_x &= -f'(x+t) + g'(x-t). \end{aligned}$$

From the consistency condition $R_{tx} = R_{xt}$ we obtain

$$f''(x+t) + g''(x-t) = 0.$$

Since $x+t$ and $x-t$ are independent variables, we have

$$f''(x+t) = \lambda, \quad g''(x-t) = -\lambda, \quad \lambda = \text{const}$$

and

$$\begin{aligned} f &= \frac{\lambda}{2}(x+t)^2 + a_1(x+t) + a_0, \\ g &= -\frac{\lambda}{2}(x-t)^2 + b_1(x-t) + b_0, \end{aligned}$$

where a_1, a_0, b_1, and b_0 are some constants. Further, we find

$$S = 2\lambda xt + a_1(x+t) + b_1(x-t) + a_0 + b_0.$$

From the system

$$\begin{aligned} R_t &= S_x = 2\lambda t + a_1 + b_1, \\ R_x &= -S_t = -2\lambda x - a_1 + b_1 \end{aligned}$$

we find

$$R = \lambda(t^2 - x^2) + a_1(t-x) + b_1(t+x) + a_0 + c_0,$$

where c_0 is some constant. Using formula (3), we find U,

$$U = \frac{4\lambda}{R} + 4\lambda^2(t^2 - x^2) + 4\lambda\big(t(a_1+b_1) + x(b_1 - a_1)\big) + 4a_1b_1.$$

Thus, the functions

$$R = \lambda(t^2 - x^2) + a_1(t - x) + b_1(t + x) + a_0 + c_0,$$
$$S = 2\lambda xt + a_1(x + t) + b_1(x - t) + a_0 + b_0,$$
$$U = \frac{4\lambda}{R} + 4\lambda^2(t^2 - x^2) + 4\lambda\big(t(a_1 + b_1) + x(b_1 - a_1)\big) + 4a_1b_1,$$

where a_1, a_0, b_1, b_0, and c_0 are some constants, are a solution of system (2).

Example 2. Let

$$S_t = \varphi(R),$$
$$S_x = \psi(R),$$

where φ and ψ are some functions of a single argument. Then from the consistency condition $S_{xt} = S_{tx}$ we obtain

$$\varphi'(R)R_x = \psi'(R)R_t.$$

Consequently, the function R is a solution of the implicit equation

$$t\varphi'(R) + x\psi'(R) = f(R),$$

where f is some function of a single argument.

Equation (36) takes the form

$$\varphi'(R)R_t - \psi'(R)R_x = \frac{2}{R}\big(\psi(R)R_x - \varphi(R)R_t\big)$$

or

$$\left(\psi'(R) + \frac{2\psi(R)}{R}\right) R_x = \left(\varphi'(R) + \frac{2\varphi(R)}{R}\right) R_t.$$

The matrix

$$\begin{pmatrix} \psi'(R) & \varphi'(R) \\ \varphi'(R) + \dfrac{2\varphi(R)}{R} & \psi'(R) + \dfrac{2\psi(R)}{R} \end{pmatrix}$$

must be singular. Hence

$$\psi'(R)\left(\psi'(R) + \frac{2\psi(R)}{R}\right) = \varphi'(R)\left(\varphi'(R) + \frac{2\varphi(R)}{R}\right).$$

Thus, the functions $\varphi(R)$ and $\psi(R)$ may not be arbitrary. They are connected by this relation. For the function U we have formula (35),

$$U = \frac{1}{R}(R_{tt} - R_{xx}) + \big(\psi^2(R) - \varphi^2(R)\big).$$

Conclusion. Let the functions $\varphi(R)$ and $\psi(R)$ be connected by the differential relation

$$\psi'^2(R) - \varphi'^2(R) = \frac{1}{R}\Big(\big(\varphi^2(R)\big)' - \big(\psi^2(R)\big)'\Big). \tag{37}$$

Then the functions R, S, and U are such that

1. R is a solution of the equation

$$t\varphi'(R) + x\psi'(R) = f(R),$$

where f is some function of a single argument,

2. S is a solution of the system

$$S_t = \varphi(R),$$
$$S_x = \psi(R),$$

which is consistent by virtue of item 1,

3.
$$U = \frac{1}{R}(R_{tt} - R_{xx}) + \big(\psi^2(R) - \varphi^2(R)\big)$$

satisfy system (2).

Let us show that the general solution of Eq. (37) is given by the formulas

$$\varphi = \frac{1}{2}\left(b + \exp\left(-\int \frac{b'}{(Rb)'}dR\right)\right),$$
$$\psi = \frac{1}{2}\left(b - \exp\left(-\int \frac{b'}{(Rb)'}dR\right)\right),$$

where $b = b(R)$ is an arbitrary function.

Indeed, let us rewrite (10) in the form

$$R\big(\varphi'^2 - \psi'^2(R)\big) + (\varphi^2 - \psi^2)' = 0,$$

or

$$R(\varphi' - \psi')(\varphi' + \psi') + \big((\varphi - \psi)(\varphi + \psi)\big)' = 0.$$

Introduce the notation

$$a = \frac{1}{2}(\varphi + \psi),$$
$$b = \frac{1}{2}(\varphi - \psi).$$

Then the equation takes the form

$$Ra'b' + a'b + ab' = 0,$$

or

$$\frac{a'}{a} + \frac{b'}{Rb' + b}.$$

By integration we find

$$a = \exp\left(-\int \frac{b'}{(Rb)'}dR\right)$$

and return to the functions φ and ψ.

CONCLUSIONS

It is apparently of interest to carry out a similar analysis of the equations relating the amplitude and potential functions and the phase function for the multidimensional Schrödinger equation, and also to apply the results obtained to nonlinear Schrödinger equations when the potential is a function of amplitude, for example, for the cubic Schrödinger equation [13, 32]. We also note that some results related to the constructive approach to the study of equations of mathematical physics and their application to the search for coefficients and solutions can be found in [15, 16, 33–36].

FUNDING

This work was carried out with financial support from the Fundamental Scientific Research Program of the Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences, project no. FWNF-2022-0009.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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