## On correspondence between right near-domains and sharply 2–transitive groups Andrey A. Simonov

a.simonov@g.nsu.ru

ABSTRACT. The right near-domain is defined to loosen neardomain axioms. Correspondence of a class of the right near-domains and a class of sharply 2–transitive groups is constructed.

Keywords: near-domain, sharply 2-transitive groups.

In [1, 2] for exposition of sharply 2-transitive groups the concept near-domain is introduced. Near-domain is an algebraic system  $(B, \cdot, +, ^{-1}, 0)$  with two binary operations  $\cdot, +$  for which axioms hold:

1. (B, +) is a loop with a unit element 0;

- 2.  $a + b = 0 \Rightarrow b + a = 0;$
- 3.  $(B_1, \cdot, {}^{-1})$  is a group with an unit element e, where  $B_1 = B \setminus \{0\}$ ;
- 4.  $(\forall x \in B) \quad x \cdot 0 = 0;$
- 5.  $(\forall x, y, z \in B)$   $(x+y) \cdot z = x \cdot z + y \cdot z;$

6.  $(\forall a, b \in B) (\exists r_{a,b} \in B_1)$   $(x+a) + b = x \cdot r_{a,b} + (a+b)$  for any  $x \in B$ .

Until recently it is not known any example of a near-domain which is not a near-field. In the given work it is offered to loosen near-domain axioms, having left only necessary ones for construction of sharply 2-transitive groups. In particular, it is offered to refuse from axioms 2, 4 and to loosen axioms 1, 5.

Let's define the right near-domain as algebraic system  $(B, \cdot, +, -, {}^{-1}, 0)$  with operations:  $(+): B \times B_1 \to B, (-): B \times B_1 \to B, (\cdot): B \times B_1 \to B$ , where  $B_1 = B \setminus \{0\}$ , for which axioms are hold

A1.  $(\forall x \in B)(\forall y \in B_1) (x - y) + y = x;$ 

- A2.  $(\forall x \in B)(\forall y \in B_1) (x+y) y = x;$
- A3.  $(\forall x \in B_1) \ x x = 0;$
- A4.  $(B_1, \cdot, -1)$  is a group with an unit element  $e \in B_1$ ;
- A5.  $(\forall x \in B)(\forall y, z \in B_1)(\exists h(y, z) \in B_1) (x + y)z = xh(y, z) + yz;$
- A6.  $(\forall x \in B)(\forall y, z \in B_1 : y + z \neq 0)(\exists r(y, z) \in B_1) (x + y) + z = xr(y, z) + (y + z);$

A7.  $(\forall x \in B)(\forall z \in B_1)(\exists v(z) \in B_1) (x + (0 - z)) + z = xv(z).$ 

Axioms A1—A3 define algebraic system (B, +, -, 0) as the right loop. We define labels L(x) = 0 - x then from A1 follows L(x) + x = 0. Thus the map  $L: B_1 \to B_1$  defines left inverse in the right loop.

Let's consider now the elementary consequences of axioms.

Lemma. In the right near-domain the following properties hold:

1.  $(\forall x \in B_1) \ 0x = 0;$ 

2. h(x,y) = EL(x)L(xy), where  $E(x) = x^{-1}$ , EL — superposition of transformations L and E;

3. r(y,z) = E(L(z) - y)L(y + z);

4. x - z = xEv(z) + L(z);

5.  $v(z) = EL^2(z)z$ 

Let's define a map  $u: B_1 \to B$  by the rule u(x) = 0x.

From A5 follows, that  $(\forall x, y \in B_1) (L(x) + x)y = L(x)h(x, y) + xy = u(y)$ , hence

$$h(x,y) = EL(x)(u(y) - xy).$$
 (1)

If we will sequentially apply A5 for arbitrary  $z, t \in B_1$  then we receive:

$$h(y,z)h(yz,t) = h(y,zt).$$

Let's write the given equality applying the expression (1). With the reduction account, we will receive equality : (u(z) - yz)EL(yz) = e, hence, u(z) = L(yz) + yz = 0. Thus, the first and second conditions of the lemma are satisfied.

Let's consider now consequences from A6. Let  $x = L(y+z)(r(y,z))^{-1} \Rightarrow (L(y+z)(r(y,z))^{-1} + y) + z = 0$ , whence we will receive the expression from the third condition of the lemma.

In the case when y + z = 0, we will consider consequences from A7 and A2: x + L(z) = xv(z) - z. We define x' = xv(z), hence the fourth condition of the lemma is fulfilled x'E(v(z)) + L(z) = x' - z.

Let's note A2 with the condition of the received expression (x + z) - z = (x + z)Ev(z) + L(z) = x. At x = 0 we will receive equality zEv(z) + L(z) = 0. Then with the account  $L^2(x) = LL(x)$ , we will come to justice of the fifth condition of the lemma.

The operation "-" is expressed through the operations "+", ".", L, E hence we will understand algebraic system  $(B, \cdot, +, -, ^{-1}, 0)$  as  $(B, \cdot, +, ^{-1}, L, 0)$ .

Let's consider the algebraic system  $(H, \cdot, \phi, {}^{-1}, 0)$  from [3], with the operations:

 $(\cdot): H \times H_1 \to H, \phi: H \to H, \text{ where } H_1 = H \setminus \{0\},\$ 

for which the following axioms are fulfilled:

F1.  $(H_1, \cdot, {}^{-1})$  is a group with an unit element e;

F2.  $0x = 0, x \in H_1;$ 

F3.  $\phi(e) = 0;$ 

F4.  $\phi(\phi(x)\phi(y)) = \phi(x\phi(y^{-1}))y, x \in H, y \in H_1 \setminus \{e_1\},$ 

The similar algebraic system was investigated in [4].

**Theorem 1.** The class of algebraic systems  $(B, \cdot, +, ^{-1}, L, 0)$  and  $(B, \cdot, ^{-1}, \phi, 0)$  are rational equivalent.

Let's introduce a map  $\phi : B \to B$ , defined in an aspect  $\phi(x) = x(0 - e) + e = xa + e$ . Let's calculate quadrate of function  $\phi$  taking into account the conditions two and five of the lemma:

$$\phi^2(x) = (xa + e)a + e = (xL(a) + a) + e = xL(a)EL^2(e) = x.$$

From the definition follows  $\phi(e) = a + e = 0$  and  $\phi(0) = e$ . By means of the map  $\phi$  it is possible to express additive operation. Really,  $\phi(x)y = (xa + e)y = xL(y) + y$ , hence, if x = zEL(y), then  $z + y = \varphi(zEL(y))y$ . Let's rewrite now

identity from A2:  $z = (z + y) - y = \phi(zEL(y))y - y$ . Having introduced labels  $t = \phi(zEL(y))y$ , we express  $z = \phi(ty^{-1})L(y)$ , then  $t - y = \phi(ty^{-1})L(y)$ .

Calculating the value t = (x + z) - (y + z) in the case  $y \neq L(z)$ , using at first A2: (x + z) = t + (y + z), and then the third identity of the lemma:  $(x + z) = (t(r(y, z))^{-1} + y) + z$ . Applying twice identity from A2, we have the identity:

$$(x+z) - (y+z) = (x-y)(L(z)-y)^{-1}L(y+z).$$

Let's rewrite the given identity with the account  $y \neq e, z = L^{-1}(e)$  replacing additive binary operations by their expressions through the function  $\phi$ :

$$\phi(\phi(x)E\phi(y)) = \phi(xy^{-1})E\phi E(y) = \phi(xy^{-1})E\phi E(y).$$
 (2)

At x = 0 this identity takes a simple form  $\phi E \phi(y) = E \phi E(y)$ , using it, we note identity (2) for  $y = E \phi E(t)$ :

$$\phi(\phi(x)\phi(t)) = \phi(x\phi E(t))t. \tag{3}$$

Thus, we have the map  $\mathbb{A}: (B, \cdot, +, {}^{-1}, L, 0) \to (B, \cdot, {}^{-1}, \phi, 0).$ 

Let's make the inverse construction. We will consider expression from F4 at  $x = e, y = t^{-1}$ , then under condition F2 and F3 we come to equality  $\varphi^2(t) = \varphi(0)t$ . On one hand  $\varphi^4(t) = (\varphi(0))^2 t$ , and on the other hand  $\varphi^4(t) = \varphi(\varphi^2(\varphi(t))) = \varphi(\varphi(0)\varphi(t))$ . It is also possible to note the last expression with the account F4 and F2:  $\varphi(\varphi(0)\varphi(t)) = \varphi(0\varphi(t^{-1}))t = \varphi(0)t$ . Thus, we come to equality  $\varphi^2(0) = \varphi(0)$ , hence,  $\varphi(0) = e$  and  $\varphi^2(t) = t$ .

From F4 for  $x = E\varphi E(y)$  follows, that  $(\forall y \in B_1 \setminus \{e\}) \varphi E\varphi(y) = E\varphi E(y)$ . By means of arbitrary bijection  $L: B_1 \to B_1$  we introduce operations

$$x + y = \varphi(xEL(y))y, \ x - y = \varphi(xy^{-1})L(y).$$

With the account of F2, F3 and  $\varphi^2 = id$  it is easy to check up the performance of the axioms A1—A3 of the right loop. The performance A5 follows from the operation definition:

$$(x+y)z = \varphi(xEL(y))yz = \varphi(xEL(y)L(yz)EL(yz))yz = xEL(y)L(yz) + yz.$$

Then we take advantage of identities  $\varphi^2 = id$ ,  $\varphi E \varphi = E \varphi E$  and F4 to receive A6:

$$\begin{split} (x+y)+z &= \varphi(\varphi(xEL(y))yEL(z))z = \\ \varphi(xEL(y)\varphi E\varphi(yEL(z)))\varphi(yEL(z))z = \\ \varphi(xEL(y)\varphi E\varphi(yEL(z))L[\varphi(yEL(z))z]EL[\varphi(yEL(z))z])\varphi(yEL(z))z = \\ xEL(y)\varphi E\varphi(yEL(z))L[\varphi(yEL(z))z] + (y+z) = \end{split}$$

 $xE(\varphi(L(z)E(y))L(y))L[\varphi(yEL(z))z] + (y+z) = xE(L(z)-y)L(y+z) + (y+z).$ Now we take advantage of identity  $\varphi^2 = id$  for construction of expression A7:

$$(x + L(z)) + z = \varphi(\varphi(xEL^2(z))L(z)EL(z))z = \varphi^2(xEL^2(z))z = xEL^2(z)z$$

For any bijection L we have constructed the map  $\mathbb{F}_L : (B, \cdot, {}^{-1}, \varphi, 0) \to (B, \cdot, +, {}^{-1}, L, 0)$  so, that the algebraic systems  $(B, \cdot, {}^{-1}, \varphi, 0)$  and  $\mathbb{A} \circ \mathbb{F}_L(B, \cdot, {}^{-1}, \varphi, 0)$  are isomorphic. In the opposite direction the algebraic systems  $(B, \cdot, +', {}^{-1}, L', 0)$  and  $\mathbb{F}_L \circ \mathbb{A}(B_1, 0, L', \cdot, +')$  are isomorphic only at L = L'.  $\Box$ 

The group  $T_2(B)$  of transformations of a set B is called sharply 2-transitive group, if for arbitrary pairs  $(x_1, x_2) \neq (y_1, y_2) \in \widehat{B^2}$ , where  $\widehat{B^2} = B^2 \setminus \{(x, x) | x \in B\}$  there exists an unique element  $g \in T_2(B)$  for which the equalities  $g(x_1) = y_1$  and  $g(x_2) = y_2$  are held.

**Theorem 2.** The class of algebraic systems  $(B, \cdot, {}^{-1}, \varphi, 0)$  and the class of sharply 2-transitive groups  $T_2(B)$  are rational equivalent. On the set  $\widehat{B^2}$  we define a function  $f: B \times \widehat{B^2} \to B$  as

$$f(x, y_1, y_2) = \varphi(x\varphi(y_1y_2^{-1}))y_2, \tag{4}$$

if  $y_2 \neq 0$  and  $f(x, y_1, 0) = xy_1$  otherwise. Not to consider two cases separately, we, by means of multiplicative partial operation  $(\cdot) : B \times B_1 \to B$ , define the groupoid on B so, that  $(\forall x \in B) x_0 = \varphi(x), 0^{-1} = 0$ .

Let's define a binary operation G on the set  $\widehat{B^2}$  in the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f(x_1, y_1, y_2) \\ f(x_2, y_1, y_2) \end{pmatrix} = \begin{pmatrix} \varphi(x_1 \varphi(y_1 y_2^{-1}))y_2 \\ \varphi(x_2 \varphi(y_1 y_2^{-1}))y_2 \end{pmatrix}.$$
(5)

Supposing, that there are pairs  $(x_1, x_2) \neq (y_1, y_2) \in \widehat{B^2}$ , that  $f(x_1, y_1, y_2) = f(x_2, y_1, y_2)$ . Then, for  $y_2 \neq 0$  after multiplication of the both parts of equality on  $y_2^{-1}$  and transformations by the function  $\varphi$ , we will come to equality  $x_1\varphi(y_1y_2^{-1}) = x_2\varphi(y_1y_2^{-1})$  from which follows, that  $x_1 = x_2$ . At  $y_2 = 0$  we get the equality  $x_1y_1 = x_2y_1$ , hence,  $x_1 = x_2$ . We have come to an inconsistency. Thus, the operation G, defined above, is a magma.

It is easy to check, that the pair  $(e, 0) \in B^2$  is the left and the right unit element. Now we check the associativity:

$$\begin{aligned} \varphi(\varphi(x_i\varphi(y_1y_2^{-1}))y_2\varphi(z_1z_2^{-1}))z_2 &= \varphi(\varphi(x_i\varphi(y_1y_2^{-1}))\varphi\varphi(y_2\varphi(z_1z_2^{-1})))z_2 = \\ \varphi(x_i\varphi(y_1y_2^{-1})\varphi E\varphi(y_2\varphi(z_1z_2^{-1})))\varphi(y_2\varphi(z_1z_2^{-1}))z_2 = \\ \varphi(x_i\varphi(y_1\varphi(z_1z_2^{-1})E\varphi(y_2\varphi(z_1z_2^{-1})))\varphi(y_2\varphi(z_1z_2^{-1}))z_2. \end{aligned}$$

We have come to a semigroup with a unit element. We will discover now the left inverse:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \varphi(x_2^{-1}) E \varphi(x_1 x_2^{-1}) \\ E \varphi(x_1 x_2^{-1}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e \\ 0 \end{pmatrix}.$$

Let's check that it is also the right inverse:

$$\varphi(x_i\varphi(\varphi(x_2^{-1})E\varphi(x_1x_2^{-1})\varphi(x_1x_2^{-1})))E\varphi(x_1x_2^{-1}) = \varphi(x_ix_2^{-1})E\varphi(x_1x_2^{-1}).$$

Thus, we defined that G is a group, but since it operates on the set  $\widehat{B^2}$  sharply transitive, then the group G at an operation on the set B it will be sharply 2-transitive, hence, we have constructed the map  $\mathbb{G} : (B, \cdot, {}^{-1}, \varphi, 0) \to T_2(B)$ .

Now we make the inverse construction and on a group  $T_2(B)$  we will construct an algebraic system  $(B, \cdot, {}^{-1}, \phi, 0)$ . For an arbitrary pair  $(e_1, e_2) \in \widehat{B^2}$  it is possible to construct the bijective map  $T_2(B) \to \widehat{B^2}$ , putting in correspondence to an element  $g \in T_2(B)$  the pair  $[x_1, x_2]$  so that

$$(e_1, e_2) \cdot g = (e_1, e_2) \cdot [x_1, x_2] = (e_1 \cdot [x_1, x_2], e_2 \cdot [x_1, x_2]) = (x_1, x_2).$$
(6)

The given bijection induces the isomorphic group  $G \simeq T_2(B)$  on the set of pairs  $\widehat{B^2}$ . The pair  $[e_1, e_2]$  is an unit of the group G.

At serial transformation of the pair  $(e_1, e_2)$  by elements  $[x_1, x_2]$  and  $[y_1, y_2]$  we come to equality:

$$[x_1, x_2][y_1, y_2] = [x_1 \cdot [y_1, y_2], x_2 \cdot [y_1, y_2]], \tag{7}$$

from which, with the account (6), follows, that on a subset  $B_1 = \{x \in B | [x, e_2] \in G\}$  it is possible to introduce the group structure naturally. The map  $e_1 \cdot [x, e_2] \mapsto x$  induces on  $B_1$  a group structure. Multiplication in the group  $B_1$ , as well as in the group  $T_2(B)$  we will write without a point. We will expand the group operation to a partial operation  $B \times B_1 \to B$ , having predetermined it in an aspect  $e_2y = e_2 \cdot [y, e_2] = e_2$  so, that  $e_2$  will be the left zero in a partial operation  $(\cdot) : B \times B_1 \to B$ .

From (6) and (7) follows,  $[e_2, e_1]$  is an involution of G. We define  $\phi : B \to B$  in an aspect  $\phi(x) = x \cdot [e_2, e_1]$ , then  $\phi(e_1) = e_2$  and

$$[e_2, e_1][x_2, x_1] = [x_1, x_2] = [\phi(x_1), \phi(x_2)][e_2, e_1].$$
(8)

For an arbitrary  $[e_1, x_2] \in G$ , at  $x_2 \in B_1 \setminus \{e_1\}$  it is possible to note:

$$[e_1, x_2] = [x_2^{-1}, e_1][x_2, e_2] = [\phi(x_2^{-1}), e_2][e_2, e_1][x_2, e_2].$$

On the other hand, with the account (8) for  $[e_1, x_2]$  it is fair

$$[e_1, x_2] = [e_2, e_1][\varphi(x_2), e_2][e_2, e_1]$$

Having taken advantage of the two received expressions and equating outcomes of transformation arbitrary  $t \in B$  by element  $[e_1, x_2] \in G$ , we come to identity:

$$\phi(\phi(t)\phi(x_2)) = \phi(t\phi(x_2^{-1}))x_2, \ t \in B, x_2 \in B_1 \setminus \{e_1\}.$$

The map  $\mathbb{F}_{(e_1,e_2)}: T_2(B) \to (B, \cdot, {}^{-1}, \phi, 0)$  is constructed, putting in correspondence to group  $T_2(B)$  algebraic system  $(B, \cdot, {}^{-1}, \phi, 0)$ .

Let's notice still, that for arbitrary  $[x_1, x_2] \in T_2(B)$  it is possible to note:

$$[x_1, x_2] = \begin{cases} [\phi(x_1 x_2^{-1}), e_2][e_2, e_1][x_2, e_2], & x_2 \in B_1, \\ [x_1, e_2], & x_2 = e_2. \end{cases}$$

Then for arbitrary  $t \in B$  under condition of  $x_2 \neq e_2$  and  $t \cdot [x_1, e_2] = tx_1$  the equality:

$$t \cdot [x_1, x_2] = t \cdot [\phi(x_1 x_2^{-1}), e_2][e_2, e_1][x_2, e_2] = \phi(t\phi(x_1 x_2^{-1}))x_2$$
(9)

is fair. Comparing (4), (5) with (9) and (7) we come to that there is a natural isomorphism  $\mathbb{G} \circ \mathbb{F}_{(e_1,e_2)} : T_2(B) \to T'_2(B)$ , thus  $\mathbb{G} \circ \mathbb{F}_{(e_1,e_2)} = id$ . Isomorphism of algebraic systems  $\mathbb{F}_{(e_1,e_2)} \circ \mathbb{G} : (B, \cdot, {}^{-1}, \phi, 0) \to (B', \cdot', {}^{-1}, \phi, e_2)$  is set by map  $\mathbb{F}_{(e_1,e_2)} \circ \mathbb{G} : x \mapsto \varphi(x\varphi(e_1e_2^{-1}))e_2$ , thus  $\mathbb{F}_{(e_1,e_2)} \circ \mathbb{G} = id$ .

Let's consider some examples of the right near-domains constructed over a skew field K for which  $\varphi(x) = -x + 1$ ,  $x \in \mathbb{K}$ . As the first example we consider L(x) = ax:

$$x \oplus y = -xa^{-1} + y, \ x \ominus y = -xa + ay, \ r(y,z) = -a^{-1}, \ v(z) = a^{-2}.$$

In such right near-domain bilaterial distributivity is fulfilled and the identity  $L(x \oplus y) = L(x) \oplus L(y)$  is hold. For the second example over a skew field we consider  $L(x) = -x^{-1}$ , then

$$x\oplus y = xy^2 + y, \ x \ominus y = xy^{-2} - y^{-1}, \ r(y,z) = y^2 z(z+y)^{-1}(yz+1), \ h(y,z) = z^{-1}.$$

For the given right loop  $L(x \oplus y) \neq L(x) \oplus L(y)$ , but it is fulfilled  $L(x) \oplus x = x \oplus L(x) = 0$ .

## References

- Karzel H. Inzidenzgruppen I. Lecture Notes by Pieper, I. and Sorensen, K., University of Hamburg (1965), 123-135.
- [2] Karzel H. Zusammenhange zwischen Fastbereichen, scharf zweifach transitiven Permutationsgruppen und 2-Strukturen mit Rechtecksaxiom, Abh. Math. Sem. Univ. Hamburg 32 (1968), 191-206.
- [3] Simonov A.A. About correspondence between neardomains and groups. Algebra i Logic. vol. 45, 2, 2006.
- [4] Leissner W. On the Functional Equation  $\phi(xy^{-1}) = \phi(\phi(x)\phi(y)^{-1})\phi(y^{-1})$ over a Group. Report of Meetings. Elfte internationale Tagung über Funktionalgleichungen in Oberwolfach vom 14. bis 20. Dezember 1973.
- [5] Maltsev A. I. Structural performance of some classes of algebras, Doklady of the Academy of Sciences of the USSR, 120, No. 1, 29-32, 1958.