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THE GEOMETRY OF SPACES OF CONSTANT CURVATURE AS A SPECIAL CASE OF THE THEORY OF PHYSICAL STRUCTURES

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As is well known, there exist several variants of the axiomatization of the structure of Euclidean geometry. In the present article we suggest some new methods of axiomatizing the structure of locally Euclidean, locally symplectic, and locally noneuclidean geometry of constant curvature. This method may be considered as a special case of the general principle of phenomenological symmetry conjectured in [1] for a unified treatment of different physical theories of phenomenological type such as mechanics, thermodynamics, special theory of relativity, and so forth.

In the present special case, the principle of phenomenological symmetry may be formulated as follows.

Let $\mathfrak{M} = \{i, k, \cdots, l, \cdots\}$ be a set of objects of arbitrary nature, which we call points, and associate a real number a_{ik} with each ordered pair (i, k), $i \neq k$. In addition* we assume one of the following two conditions:

I_s. For each pair (i, k), $i \neq k$, $a_{ik} = a_{ki}$.

I_a. For each pair (i, k), $i \neq k$, $a_{ik} = -a_{ki}$.

Let $\mathfrak{M}_n = \{i_1, i_2, \cdots, i_n\}$ be an arbitrary n-element subset of $\mathfrak{M}$. Associated with it is a system $\mathfrak{A}_n$ of $\frac{1}{2}n(n-1)$ numbers

$$\mathfrak{A}_n = \{a_{i_r i_s}\}$$

where r and s run through all possible values such that $1 \leq r < s \leq n$.

The system $\mathfrak{A}_n$ is contained in the $\frac{1}{2}n(n-1)$-dimensional vector space $R^{n(n-1)/2}$. We denote by $\mathfrak{C}_n$ the set of all points in $R^{n(n-1)/2}$ occurring as the image of all possible n element subsets of $\mathfrak{M}_n$.

We introduce the following fundamental condition:

II. The set $\mathfrak{C}_n$ is represented by a $\frac{1}{2}n(n-1)-1$-dimensional manifold of class C^∞.

Condition II constitutes the basis of the principle of phenomenological symmetry suitable for the case under consideration. If the functions a_{ik} $(i \neq k)$ satisfy conditions I and II we shall say that the set $\mathfrak{M}$ is given a physical structure of rank n.

We introduce the concept of equivalent physical structures. Two physical structures of the same rank n with functions a_{ik} and b_{ik} will be called equivalent if there exists a monotonic function $\chi(x)$ of one variable such that for arbitrary i and k we have $b_{ik} = \chi(a_{ik})$. (We note that in a description of experimental validity, the arbitrary choice of the function $\chi(x)$ may be interpreted as an arbitrary choice of scale on a measuring device.)

* It may be shown that the condition "either $a_{ik} = a_{ki}$ or $a_{ik} = -a_{ki}$" is too strong; it is sufficient to require that $a_{ik} = f(a_{ki})$, where $f(x)$ is an a priori unknown function.

The problem consists of finding (for $n \geq 2$) functions a_{ik} satisfying conditions in addition to those stated above.

As is shown in the preliminary analysis [2], the requirement of phenomenological symmetry is very rigid in the sense that all possible physical structures of a given rank are quite limited.

It may be shown [3] that for well-known locally Euclidean geometries, locally noneuclidean geometries with constant curvature, and locally symplectic geometries, the principle of phenomenological symmetry holds. (An n-dimensional geometry realizes the appropriate physical structure of rank $n+2$, if we take a_{ik} to be the distance between elements i and k.) The manifold of the geometry is given up to equivalence by the following equations:

I_s. $a_{ik} = a_{ki}$

1)

$$\underbrace{\begin{vmatrix} a & a_{ik} & a_{il} & \cdots & a_{im} \\ a_{ik} & a & a_{kl} & \cdots & a_{km} \\ a_{il} & a_{kl} & a & \cdots & a_{lm} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{lm} & a_{km} & a_{lm} & \cdots & a \end{vmatrix}}_{n} = 0$$

or in parametric form

$$a_{ik} = g^0_{\sigma\rho}\, x^\sigma(i)\, x^\rho(k), \quad \sigma, \rho = 1, 2, \ldots, n-1,$$

where the $x^\sigma(i)$ ($\sigma = 1, 2, \cdots, n-1$) are $(n-1)$ real parameters involving the point i and the relationship between it and the single equation

$$g^0_{\rho\sigma}\, x^\rho(i)\, x^\sigma(i) = a$$

where a is an arbitrary constant;

2)

$$g^0_{\rho\sigma} = \begin{cases} 0, & \rho \neq \sigma, \\ \pm 1, & \rho = \sigma. \end{cases}$$

$$\underbrace{\begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & a_{ik} & a_{il} & \cdots & a_{im} \\ 1 & a_{ik} & 0 & a_{kl} & \cdots & a_{km} \\ 1 & a_{il} & a_{kl} & 0 & \cdots & a_{lm} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{im} & a_{km} & a_{lm} & \cdots & 0 \end{vmatrix}}_{n} = 0$$

or in parametric form

$$a_{ik} = g^0_{\mu\nu}\,(x^\mu(i) - x^\mu(k))\,(x^\nu(i) - x^\nu(k)), \quad \mu, \nu = 1, 2, \ldots, n-2,$$

where the $x^\mu(i)$ ($\mu = 1, 2, \cdots, n-2$) are $(n-2)$ arbitrary real parameters involving the point i.

I_a. $a_{ik} = -a_{ki}$.

1) $n = 2p - 1$,

$$\underbrace{\begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & a_{ik} & a_{il} & \cdots & a_{im} \\ -1 & -a_{ik} & 0 & a_{kl} & \cdots & a_{km} \\ -1 & -a_{il} & -a_{kl} & 0 & \cdots & a_{lm} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -a_{im} & -a_{km} & -a_{lm} & \cdots & 0 \end{vmatrix}}_{n} = 0$$

or in parametric form:

$$a_{ik} = x^0(i) - x^0(k) + x^\nu(i) y_\nu(k) + x^\nu(k) y_\nu(i) \qquad \nu = 1, 2, \dots, p-2,$$

where the $x^0(i)$, $x^\nu(i)$, $y_\nu(i)$ ($\nu = 1, 2, \cdots, p-2$) are $(n-2)$ arbitrary real parameters involving the point i.

2) $n = 2p$

$$\underbrace{\begin{vmatrix} 0 & a_{ik} & a_{il} & \cdots & a_{im} \\ -a_{ik} & 0 & a_{kl} & \cdots & a_{km} \\ -a_{il} & -a_{kl} & 0 & \cdots & a_{lm} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{im} & -a_{km} & -a_{lm} & \cdots & 0 \end{vmatrix}}_{n} = 0$$

or in parametric form:

$$a_{ik} = x^\lambda(i) y_\lambda(k) - x^\lambda(k) y_\lambda(i), \qquad \lambda = 1, 2, \dots, p-1,$$

where the $x^\lambda(i)$, $y_\lambda(i)$ ($\lambda = 1, 2, \cdots, p-1$) are $(n-2)$ arbitrary real parameters involving the point i.

It is easy to see that the manifold $\mathfrak{C}_n$ in the form I_s (1) corresponds to an $(n-2)$-dimensional space of (positive or negative) constant curvature $1/a$; the manifold $\mathfrak{C}_n$ in the form I_s (2) is an $(n-2)$-dimensional Euclidean (or pseudo-Euclidean) space; finally a $\mathfrak{C}_n$ in the form I_a (1) or I_a (2) is an $(n-2)$-dimensional symplectic space.

Therefore all the manifolds enumerated above are special cases of manifolds realized as physical structures of rank n. (We shall call these manifolds phenomenological spaces of dimension $n-2$.)

The question arises: Are the symplectic spaces and the spaces of positive, zero, or negative curvature the only phenomenological spaces?

For the case $n = 3$, an exhaustive solution to this question has been obtained by G. G. Mihaĭličenko. He showed that all phenomenological structures of dimension 1 are equivalent to the Euclidean line.

But concerning the cases where $n \geq 4$, the question of the existence of new types of phenomenological spaces, different from Euclidean, symplectic, or constant curvature spaces remains open.

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