Dokl. Akad. Nauk SSSR
Tom 206 (1972), No. 5

Soviet Math. Dokl.
Vol. 13 (1972), No. 5

THE SOLUTION OF FUNCTIONAL EQUATIONS IN THE THEORY OF PHYSICAL STRUCTURES

UDC 517.948

G. G. MIHAĬLIČENKO*

In [1] Ju. I. Kulakov gave a mathematical formulation of the theory of physical structures. In that paper the simplest case was considered, using a method of parametrization. However, in application to structures of higher rank this method encounters considerable difficulties. On the other hand, the statement of the original axioms in [4] was given with regard to the parametric method of investigation used. In this paper a more natural equivalent statement of the axioms of a physical structure is given and the solution of functional equations arising in this theory is presented. Methods for solution of the functional equations were proposed by the author in the consideration of a ternary physical structure [2].

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two sets, whose elements will be denoted by lower-case Latin and Greek letters, respectively, and let $a\colon \mathfrak{M} \times \mathfrak{N} \to R$ be a real function assigning to each pair $\langle i, \alpha\rangle$ in $\mathfrak{M} \times \mathfrak{N}$ a number $a_{i\alpha} \in R$. For convenience we shall write (i, α) in place of $a_{i\alpha}$ in the sequel. Two elements $i, j \in \mathfrak{M}$ will be regarded as equivalent, denoted by $i \sim j$, if $(i, \alpha) = (j, \alpha)$ for every $\alpha \in \mathfrak{N}$. We shall assume that all equivalent elements in $\mathfrak{M}$ are identified, i.e. if $i \sim j$, then i and j coincide: $i = j$. If there is at least one α such that $(i, \alpha) \neq (j, \alpha)$, then i and j will be assumed to be different: $i \neq j$. Coincidence and noncoincidence of elements $\alpha, \beta \in \mathfrak{N}$ are defined analogously.

We define topologies on the sets $\mathfrak{M}$ and $\mathfrak{N}$ by introducing fundamental systems of neighborhoods. Let $i_0 \in \mathfrak{M}$ be a fixed element, which we shall sometimes call a point, and let $\epsilon > 0$. We denote by $P(i_0, \epsilon)$ the totality of all those elements $i \in \mathfrak{M}$ for which $|(i, \alpha) - (i_0, \alpha)| < \epsilon$ for every $\alpha \in \mathfrak{N}$. The family of all sets $P(i_0, \epsilon)$ for all possible value of the positive number ϵ is taken as a fundamental system of neighborhoods of i_0. An arbitrary neighborhood $P(i_0)$ of i_0 is a subset of $\mathfrak{M}$ containing some neighborhood of the fundamental system. Analogously we introduce the neighborhood $Q(\alpha_0, \epsilon)$ and a fundamental system of neighborhoods of $\alpha_0 \in \mathfrak{N}$. We shall denote an arbitrary neighborhood of α_0 by $Q(\alpha_0)$. It is easy to see that the systems $P(i_0)$ and $Q(\alpha_0)$ satisfy the neighborhood axioms and determine unique topological structures on the sets $\mathfrak{M}$ and $\mathfrak{N}$. $\mathfrak{M} \times \mathfrak{N}$ is then given the product topology derived from the topologies on $\mathfrak{M}$ and $\mathfrak{N}$. Note that the function $a\colon \mathfrak{M} \times \mathfrak{N} \to R$ introduced above is continuous relative to the product topology. An element of a cartesian product will be called a cortege.

AMS (MOS) subject classifications (1970). Primary 39A30, 70G99.

**Editor's note*. The present translation incorporates suggestions made by the author.

Let $\mathfrak{M}^n = \mathfrak{M} \times \mathfrak{M} \times \cdots \times \mathfrak{M}$ and $\mathfrak{N}^n = \mathfrak{N} \times \mathfrak{N} \times \cdots \times \mathfrak{N}$ be the m-and n-fold cartesian products of the sets $\mathfrak{M}$ and $\mathfrak{N}$, where $m \geq n \geq 2$ are integers. We construct a function $a^{(m,n)}$: $\mathfrak{M}^m \times \mathfrak{N}^n \to R^{mn}$ by assigning to each cortege $\langle i, j, k, \cdots, v, \alpha, \beta, \gamma, \cdots, \tau\rangle$ of length $m+n$ and $m \times n$ matrix of numbers

$$(i, j, k, \cdots, v, \alpha, \beta, \gamma, \cdots, \tau) = \begin{Vmatrix} (i, \alpha) & (j, \alpha) \dots (v, \alpha) \\ (i, \beta) & (j, \beta) \dots (v, \beta) \\ \dots & \dots \\ (i, \tau) & (j, \tau) \dots (v, \tau) \end{Vmatrix}, \tag{1}$$

considered as a point in the mn-dimensional space R^{mn}. In order to shorten the notation it is convenient to write the matrix (1) in one line: $(i, j, k, \cdots, v, \alpha, \beta, \gamma, \cdots, \tau) \in R^{mn}$. Transformations corresponding to permutations of rows or columns of (1) we shall call canonical permutations of the space R^{mn}. Points of R^{mn} passing into one another under a canonical permutation will be called canonically conjugate. We denote the set of values of the function $a^{(m,n)}$: $\mathfrak{M}^m \times \mathfrak{N}^n \to R^{mn}$ by N. Note that the set N is invariant relative to canonical permutations. A cortege in a cartesian product is considered to be nondiagonal if all of its elements from one set are different.

We shall say that a binary physical structure of rank (m, n) is given on the sets $\mathfrak{M}$ and $\mathfrak{N}$ if the following conditions hold:

A. The mapping $a[\beta, r, \cdots, \tau]$: $\mathfrak{M} \to R^{n-1}$ defined by the function $i \to (i, \beta, \gamma, \cdots, \tau) \in R^{n-1}$ is open for every nondiagonal cortege $\langle\beta, \gamma, \cdots, \tau\rangle \in \mathfrak{N}^{n-1}$, where $\beta \neq \gamma \neq \cdots \neq \tau$; the mapping $a[j, k, \cdots, v]$: $\mathfrak{N} \to R^{m-1}$ defined by the function $\alpha \to (j, k, \cdots, v, \alpha) \in R^{m-1}$ is open for every nondiagonal cortege $\langle j, k, \cdots, v\rangle \in \mathfrak{M}^{m-1}$, where $j \neq k \neq \cdots \neq v$.

B. There exists an analytic function Φ: $R^{mn} \to R$ such that the set determined by $\Phi = 0$ coincides with N, i.e. $M = N$ and

$$\Phi[(i, j, k, \cdots, v, \alpha, \beta, \gamma, \cdots, \tau)] = 0 \tag{2}$$

for every cortege $\langle i, j, k, \cdots, v, \alpha, \beta, \gamma, \cdots, \tau\rangle \in \mathfrak{M}^m \times \mathfrak{N}^n$.

C. The gradient of Φ is different from zero everywhere on M with the possible exception of a set of measure zero relative to M.

Condition A actually says that $\mathfrak{M}$ is at least $(n-1)$-dimensional, while $\mathfrak{N}$ is at least $(m-1)$-dimensional. This condition is somewhat stronger than the requirement that the sets $\mathfrak{M}$ and $\mathfrak{N}$ be manifolds of corresponding dimension. Interiority of the mappings $a[\beta, \gamma, \cdots, \tau]$: $\mathfrak{M} \to R^{n-1}$ and $a[j, k, \cdots, v]$: $\mathfrak{N} \to R^{m-1}$ imposes certain restrictions on the original function a: $\mathfrak{M} \times \mathfrak{N} \to R$. In this connection the topology defined on the sets $\mathfrak{M}$ and $\mathfrak{N}$ does not degenerate to the discrete topology. In general, it might have been possible to assume at once that $\mathfrak{M}$ and $\mathfrak{N}$ are topological spaces and the function a: $\mathfrak{M} \times \mathfrak{N} \to R$ is continuous, without troubling oneself with the definition of a topology, but this would have increased the number of initial assumptions. The basic content of the theory of physical structures consists in condition B, which arose from an analysis of the structure of certain physical laws, if they are written in a form containing only quantities measured by experiment [3]. Condition C means that the singular points on M constitute at most a set of measure zero relative to M.

Theorem. *If the triple* $\langle \mathfrak{M}, \mathfrak{N}, a: \mathfrak{M} \times \mathfrak{N} \to R \rangle$ *forms a binary physical structure of rank* (m, n), *then in any neighborhoods* $P(i_0)$, $Q(\alpha_0)$ *of arbirtary points* $i_0 \in \mathfrak{M}$, $\alpha_0 \in \mathfrak{N}$ *there exist elements* $i_1 \in P(i_0)$, $\alpha_1 \in Q(\alpha_0)$ *in some neighborhoods* $P(i_1)$, $Q(\alpha_1)$ *of which the function* $a: P(i_1) \times Q(\alpha_1) \to R$ *and the set* $N_1 \subset N$ *of values of the function* $a^{(m,n)}: A[P(i_1)]^m \times [Q(\alpha_1)]^n \to R^{mn}$ *can be given in the following way:*

a) *for* $m = n = 2$,

$$(i, \alpha) = \Psi^{-1}(x_i + \xi_\alpha),$$

$$\Psi[(i, \alpha)] - \Psi[(i, \beta)] - \Psi[(j, \alpha)] + \Psi[(j, \beta)] = 0; \tag{3}$$

b) *for* $m = n + 2 = 4$,

$$(i, \alpha) = \Psi^{-1}[(x_i\xi_\alpha^1 + \xi_\alpha^2) / (x_i + \xi_\alpha^3)],$$

$$\begin{vmatrix} \Psi[(i, \alpha)] & \Psi[(i, \beta)] & \Psi[(i, \alpha)] & \Psi[(i, \beta)] & 1 \\ \Psi[(j, \alpha)] & \Psi[(j, \beta)] & \Psi[(j, \alpha)] & \Psi[(j, \beta)] & 1 \\ \Psi[(k, \alpha)] & \Psi[(k, \beta)] & \Psi[(k, \alpha)] & \Psi[(k, \beta)] & 1 \\ \Psi[(l, \alpha)] & \Psi[(l, \beta)] & \Psi[(l, \alpha)] & \Psi[(l, \beta)] & 1 \end{vmatrix} = 0; \tag{4}$$

c) *for* $m = n \geq 3$,

$$(i, \alpha) = \Psi^{-1}(x_i^1\xi_\alpha^1 + \ldots + x_i^{m-2}\xi_\alpha^{m-2} + x_i^{m-1}\xi_\alpha^{m-1}),$$

$$\begin{vmatrix} \Psi[(i, \alpha)] & \Psi[(i, \beta)] & \ldots & \Psi[(i, \tau)] \\ \Psi[(j, \alpha)] & \Psi[(j, \beta)] & \ldots & \Psi[(j, \tau)] \\ \ldots & \ldots & \ldots & \ldots \\ \Psi[(v, \alpha)] & \Psi[(v, \beta)] & \ldots & \Psi[(v, \tau)] \end{vmatrix} = 0, \tag{5}$$

and also

$$(i,\alpha) = \Psi^{-1}(x_i^1\xi_\alpha^1 + \ldots + x_i^{m-2}\xi_\alpha^{m-2} + x_i^{m-1} + \xi_\alpha^{m-1}),$$

$$\begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & \Psi[(i, \alpha)] & \Psi[(i, \beta)] & \ldots & \Psi[(i, \tau)] \\ 1 & \Psi[(j, \alpha)] & \Psi[(j, \beta)] & \ldots & \Psi[(j, \tau)] \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \Psi[(v, \alpha)] & \Psi[(v, \beta)] & \ldots & \Psi[(v, \tau)] \end{vmatrix} = 0; \tag{6}$$

d) *for* $m = n + 1 \geq 3$,

$$(i, \alpha) = \Psi^{-1}(x_i^1\xi_\alpha^1 + \ldots + x_i^{m-2}\xi_\alpha^{m-2} + \xi_\alpha^{m-1}),$$

$$\begin{vmatrix} 1 & \Psi[(i, \alpha)] & \Psi[(i, \beta)] & \ldots & \Psi[(i, \tau)] \\ 1 & \Psi[(j, \alpha)] & \Psi[(j, \beta)] & \ldots & \Psi[(j, \tau)] \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \Psi[(v, \alpha)] & \Psi[(v, \beta)] & \ldots & \Psi[(v, \tau)] \end{vmatrix} = 0, \tag{7}$$

where $i, j, k, \cdots, v \in P(i_1)$, $\alpha, \beta, \gamma, \cdots, \tau \in Q(\alpha_1)$, ψ *is a strictly monotone analytic function of one variable defined in some neighborhood of the point* $(i_1\ \alpha_1) \in R$, ψ^{-1} *is the inverse function, and* x_i, ξ_α *are independent parameters given by arbitrary open mappings* $x: \mathfrak{M} \to R^{n-1}$, $\xi: \mathfrak{N} \to R^{m-1}$;

e) *for* $m - n \geq 2$, *except in the case* $m = n + 2 = 4$, *binary physical structures do not exist.*

The proof of this theorem is rather tedious and constitutes the content of the author's dissertation. Some very simple cases were considered in [4]. We remark that the result obtained was local. The possibility of its analytic continuation to the whole set N has not yet been determined. The restriction $m \geq n$ is obviously not essential. It is easy to rewrite the theorem for the case $m \leq n$. Kulakov [3] assumed the existence of solutions of (5) and (7). Solutions of (4) and (6) were first found by the author [4].

By an analogous scheme it is possible to introduce axioms for ternary physical structures of rank (m, n, l) on three sets $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{L}$. However, preliminary studies show that ternary structures exist only in the simplest case $m = n = l = 2$. In addition, the set N of values of the function $a^{(m,n,l)}$: $\mathfrak{M}^m \times \mathfrak{N}^n \times \mathfrak{L}^l \to R^{mnl}$ can be given locally by the equation

$$\Psi[(i, \alpha, \mu)] - \Psi[(i, \alpha, \nu)] - \Psi[(i, \beta, \mu)] + \Psi[(i, \beta, \nu)] - \Psi[(j, \alpha, \mu)]$$
$$+ \Psi[(j, \alpha, \nu)] + \Psi[(j, \beta, \mu)] - \Psi[(j, \beta, \nu)] = 0,$$

where (i, α, μ) is the value of the function a: $\mathfrak{M} \times \mathfrak{N} \times \mathfrak{L} \to R$. For all remaining values of the integers m, n, l ternary physical structures of rank (m, n, l) do not exist. At the same time, the binary structures considered in this paper exist in other than the simplest case $m = n = 2$. Perhaps the explanation of this difference must be sought in the lack of a substantial theory of three-dimensional determinants. On the other hand, we note that the determinants (5) and (6) have the structure of Gram determinants and Cayley-Menger determinants. Under certain additional conditions, relations (5) and (6) can be used to construct linear spaces and spaces of constant curvature, in particular euclidean space [5]. In [6] Blumenthal used relations of the type (5) and (6) to construct euclidean and spherical spaces. In the theory of physical structures these relations arise as consequences of certain general symmetry principles given by conditions A, B, C, especially condition B.

The author would like to express profound gratitude to Professor Ju. G. Rešetnjak for numerous helpful remarks and discussions, resulting in a clearer formulation of the original axioms of a physical structure.

Novosibirsk State University
Received 11/SEPT/70

BIBLIOGRAPHY

1. Ju. I. Kulakov, *Mathematical formulation of a theory of physical structures*, Sibirsk. Mat. Ž. 12 (1971), 1142–1145. (Russian)

2. G. G. Mihaĭličenko, *Ternary physical structures of rank* (3, 2), Ukrain. Mat. Ž. 6 (1970), 837–841. (Russian)

3. ———, *Elements of the theory of physical structures*, Novosibirsk. Gos. Univ., Novosibirsk, 1969. (Russian) MR 42 #1379.

4. ———, Mathematical Appendix to [3].

5. Ju. I. Kulakov, *The geometry of spaces of constant curvature as a special case of the theory of physical structures*, Dokl. Akad. Nauk SSSR **193** (1970), 985–987 = Soviet Math. Dokl. 11 (1970), 1055–1057. MR 42 #5149.

6. L. M. Blumenthal, *Theory and applications of distance geometry*, Clarendon Press, Oxford, 1953. MR 14, 1009.

Translated by B. SILVER