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TWO-DIMENSIONAL GEOMETRIES

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It is well known that for any four points in the Euclidean plane there exists a relationship between the six corresponding distances. From the geometric viewpoint this means that the volume of a tetrahedron with vertices in the plane is zero. In works on distance geometry (see, for example, [1]) it is shown that if in a metric space $\mathfrak{M}$ for any four points the distances between them satisfy the same relationship as the one satisfied in the Euclidean plane then $\mathfrak{M}$ is isometric to a set of points on that plane. Similar facts are also true for planes in spaces of constant curvature (that is, in Lobachevsky and spherical spaces).

The main result of this note consists of showing that if for any four points in a space there is a relationship between the corresponding six distances, then the geometry of such a space is quite rigidly determined even when the particular type of the relationship has not been specified. Besides the geometry of planes, in spaces of constant curvature there also exist seven geometries satisfying the same condition. The conditions imposed on the relation between the distances and the metric reduce basically to differentiability of functions and independence of certain equations. The result is an application of the principle of phenomenological symmetry suggested by Kulakov [2].

The problem treated in this note was implicitly mentioned by Helmholtz in his famous paper [3]. Indeed, Helmholtz suggested that the geometry of n-dimensional space is defined by existence of bodies which can be moved by rigid motions with $n(n+1)/2$ degrees of freedom. Then for any $n+2$ points of a rigid body there must exist a dependence between all corresponding distances, since otherwise the number of degrees of freedom is reduced by one. In a plane, rigid bodies can move with three degrees of freedom, and there is an equation relating the six distances between any four points.

Now we will give the exact formulations. Let $\mathfrak{M}$ be an arbitrary set and $a(x, y)$ a real function defined for any x and y from $\mathfrak{M}$. The function a can be viewed as a metric in $\mathfrak{M}$, but the usual axioms of the metric are not required to be satisfied. However, the following condition is assumed:

1) if $x, y \in \mathfrak{M}$ and, for any $z \in \mathfrak{M}$, $a(x, z) = a(y, z)$ and $a(z, x) = a(z, y)$, then $x = y$.

We introduce in $\mathfrak{M}$ the weakest topology in which a is continuous. For $x \in \mathfrak{M}$ we denote by $P(x)$ a neighborhood of x in this topology which is Hausdorff because of condition 1).

Construct a mapping $A\colon \mathfrak{M}^4 \longrightarrow R^6$ putting in correspondence to any four points $(x, y, z, t) \in \mathfrak{M}^4$ the point

$$\langle a(x, y), a(x, z), a(x, t), a(y, z), a(y, t), a(z, t) \rangle$$

in R^6. Let $N = A(\mathfrak{M}^4)$ be the collection of all points in R^6 obtained in such a way. We will also assume that the following conditions are satisfied:

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2) The set of pairs $(x, y) \in \mathfrak{M}^2$ for which the maps a_{xy}: $z \in \mathfrak{M} \longrightarrow \langle a(z, x), a(z, y)\rangle \in R^2$ and $\tilde{a}_{xy}$: $z \in \mathfrak{M} \longrightarrow \langle a(x, z), a(y, z)\rangle \in R^2$ is open and dense in $\mathfrak{M}^2$.

3) There exists a function Φ: $\mathcal{E} \longrightarrow R$ of class C^4 defined in a neighborhood $\mathcal{E} \subset R^6$ such that N is an open subset of the set defined by the equation $\Phi = 0$; that is,

$$\Phi(a(x, y),\ a(x, z),\ a(x, t),\ a(y, z),\ a(y, t),\ a(z, t)) = 0$$

for any four points $(x, y, z, t) \in \mathfrak{M}^4$.

4) The map A: $\mathfrak{M}^4 \longrightarrow N$ is open relative to the topology in N induced from R^6.

5) The subset of N consisting of points where the first derivatives of Φ are different from zero is dense in N.

Condition 1) means that we consider only those properties of the space $\mathfrak{M}$ which can be expressed by means of the function a. Condition 2) guarantees that $\mathfrak{M}$ is a two-dimensional manifold. In fact, this condition is weaker than one needs to assure that $\mathfrak{M}$ is two-dimensional. Condition 3) is basic, since it expresses the principle of phenomenological symmetry in the general scheme of physical structures. This principle was suggested by Kulakov [4] as a way to classify laws of physics. Condition 3) is the requirement that the six distances between any four points must be dependent. Conditions 4) and 5) mean that the map A is nondegenerate. Namely, condition 4) assures that for any neighborhood in $\mathfrak{M}^4$ its image under A has dimension no smaller than N. Condition 5) means, roughly speaking, that there exist sets of four points which are in a general position.

DEFINITION. We say that a function a: $\mathfrak{M} \times \mathfrak{M} \longrightarrow R$ *defines on* $\mathfrak{M}$ *a two-dimensional distance geometry* (a physical structure) *of rank* 4 if conditions 1)–5) are satisfied.

Two geometries defined on $\mathfrak{M}$ by functions a and b are considered *equivalent* if $b(x, y) = \psi(a(x, y))$, where ψ is a function of one variable.

THEOREM. *If a function* a: $\mathfrak{M} \times \mathfrak{M} \longrightarrow R$ *defines on the set* $\mathfrak{M}$ *a two-dimensional geometry of rank* 4, *then for a set of pairs* (x, y) *dense in* $\mathfrak{M} \times \mathfrak{M}$ *and some neighborhoods* $P(x) \times P(y)$ *it is possible to introduce coordinates* x_1, x_2 *and* y_1, y_2 *such that in these coordinates the metric* $a(x, y)$ *is defined (up to the equivalence class) by one of the following expressions*:

(1) $a(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2$;

(2) $a(x, y) = \cos x_2 \cos y_2 \cos(x_1 - y_1) + \sin x_2 \sin y_2$;

(3) $a(x, y) = \sinh x_2 \sinh y_2 \cos(x_1 - y_1) - \cosh x_2 \cosh y_2$

(4) $a(x, y) = (x_1 - y_1)^2 - (x_2 - y_2)^2$;

(5) $a(x, y) = \cosh x_2 \cosh y_2 \cos(x_1 - y_1) - \sinh x_2 \sinh y_2$;

(6) $a(x, y) = x_1 y_2 - x_2 y_1$;

(7) $a(x, y) = (x_1 - y_1)^\alpha (x_2 - y_2)^\beta$;

(8) $a(x, y) = (x_1 - y_1)/(x_2 - y_2) + \ln(x_2 - y_2)$;

(9) $a(x, y) = \ln((x_1 - y_1)^2 + (x_2 - y_2)^2) + \gamma \operatorname{arctg}((x_1 - y_1)/(x_2 - y_2))$;

(10) $a(x, y) = ((x_1 - y_1)^2 + \delta_x x_2^2 + \delta_y y_2^2)/x_2 y_2$,

where $\alpha \neq 0$, $\beta \neq 0$, $\alpha \neq \beta$ *and* $\gamma \neq 0$; δ_x *and* δ_y *are constants depending on the neighborhoods* $P(x)$ *and* $P(y)$, *and* $\delta_x, \delta_y = 0, +1, -1$.

The expressions (1)–(7) define the metrics of well-known geometries: the Euclidean plane (1), the two-sphere (2), the Lobachevsky plane (3), the Minkowski plane (4), a two-dimensional hyperboloid of one sheet (5), the symplectic plane (6), and the simplicial plane (7). The metrics defined by (8)–(10), apparently, have not been considered before. Note that (10) defines a metric on a nonconnected set $\mathfrak{M}$, and on the connected components of it we have either a symplectic plane ($\delta_x = \delta_y = 0$), or a Lobachevsky plane ($\delta_x = \delta_y = +1$), or a two-dimensional hyperboloid of one sheet ($\delta_x = \delta_y = -1$).

Similarly one can introduce a system of axioms for n-dimensional distance geometry of rank $n + 2$. However, so far only one-dimensional [5] and two-dimensional geometries have been investigated. There are some reasons to expect that for $n \geqslant 3$ one can have only metrics which generalize expressions (1)–(6) and (10), while the metrics (7)–(9) are specific for two-dimensional geometry. Let us note that symplectic spaces of odd dimension have appeared, and metrics on such spaces cannot be introduced using the linear space [2]. For example, the metric of three-dimensional symplectic space is defined by expression

$$a(x, y) = x_1 y_2 - x_2 y_1 + x_3 - y_3.$$

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