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ON GROUP AND PHENOMENOLOGICAL SYMMETRIES IN GEOMETRY

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The geometry of metric spaces gives an example of a binary structure on a set $\mathfrak{M}$. The specification of a metric, understood in a very general sense as some function a: $\mathfrak{M} \times \mathfrak{M} \to R$, defines a geometry on the space $\mathfrak{M}$. From a known metric we can find the complete group of transformations of $\mathfrak{M}$ with respect to which the metric is a two-point invariant. Group symmetry lies at the basis of Klein's "Erlange program" (1872), according to which geometry is the theory of invariants of a given group of transformations of $\mathfrak{M}$ [1]. If the two-point invariant is unique in some sense, then the metric and group representations of the geometry are equivalent. On the other hand, the so-called phenomenological symmetry (to which Kulakov first directed special attention [2]) manifests itself in the geometry. In the given case the essence of phenomenological symmetry, which has become the main principle of Kulakov's theory of physical structures [3], amounts to the fact that there is a functional connection in the space between all the distances for a specified number of arbitrary points. In this article we establish that the group and phenomenological symmetries are equivalent in an n-dimensional distance geometry (see Theorem 3).

Helmholtz, in his article "Über die Thatsachen, die der Geometrie zum Grunde liegen" [4], conjectured that a metric in an n-dimensional space cannot be arbitrary if rigid bodies move with $n(n+1)/2$ degrees of freedom in the space. But then there must be a connection between all the mutual distances for any $n+2$ points of a rigid body, because in the absence of such a connection the number of degrees of freedom of an $(n+2)$-point rigid simplex with points in general position is decreased precisely by 1. Therefore, we can assume that phenomenological symmetry of an n-dimensional space is impossible for an arbitrary metric. This was shown by the author in [5] and [6] for the one-dimensional and two-dimensional cases.

We proceed from the nonrigorous discussion above to precise formulations.

Suppose that $\mathfrak{M}$ is an arbitrary set with points denoted by lower case Latin letters, and let a: $\mathfrak{M} \times \mathfrak{M} \to R$ be a function assigning to an ordered pair $\langle ij \rangle \in \mathfrak{M} \times \mathfrak{M}$ some real number $a(ij) \in R$. In some cases the domain $\mathfrak{S}_a$ of a may not coincide with the whole direct product $\mathfrak{M} \times \mathfrak{M}$, i.e., not every pair $\langle ij \rangle \in \mathfrak{M} \times \mathfrak{M}$ is assigned a number. However, we shall not mention this in what follows, understanding that a pair $\langle ij \rangle$ is always taken in the domain $\mathfrak{S}_a$. The function a: $\mathfrak{M} \times \mathfrak{M} \to R$ can be regarded as a kind of metric in $\mathfrak{M}$, although it will not be required to satisfy the usual metric axioms: symmetry, the triangle inequality, etc. We shall assume the condition

I. There exists a finite basis set $\mathfrak{M}_B$ such that if two arbitrary points $i, j \in \mathfrak{M}$ are distinct, then there is a $k \in \mathfrak{M}_B$ for which either $a(ik) \neq a(jk)$ or $a(ki) \neq a(kj)$.

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The meaning of this condition is, first of all, that only the properties of the space $\mathfrak{M}$ which can be expressed by means of the function a are being considered. The finiteness of $\mathfrak{M}_B$ in some sense indicates the finite dimensionality of $\mathfrak{M}$. Finally, if $\mathfrak{M}$ contains more than two distinct points, then, by condition I, for any $i \in \mathfrak{M}$ there is a $k \in \mathfrak{M}_B$ such that either the pair $\langle ik \rangle$ or the pair $\langle ki \rangle$ belongs to the domain of a.

DEFINITION 1. A set $\tilde{\mathfrak{M}} \subset \mathfrak{M}$ is said to be *bounded* if the ranges of $a\colon \tilde{\mathfrak{M}} \times \mathfrak{M}_B \to R$ and $a\colon \mathfrak{M}_B \times \tilde{\mathfrak{M}} \to R$ are bounded.

It is clear that every finite set is bounded, and the union of finitely many bounded sets is bounded.

We can introduce a topology on $\mathfrak{M}$ in a natural way. Let $i \in \mathfrak{M}$ be an arbitrary point, let $\tilde{\mathfrak{M}}$ be a bounded set containing the basis $\mathfrak{M}_B$, and let $\varepsilon > 0$. Denote by $P(i, \tilde{\mathfrak{M}}, \varepsilon)$ the set of all points $i' \in \mathfrak{M}$ for which $|a(i'k) - a(ik)| < \varepsilon$ and $|a(ki') - a(ki)| < \varepsilon$ for any $k \in \tilde{\mathfrak{M}}$ such that either the pair $\langle ik \rangle$ or the pair $\langle ki \rangle$ belongs to the domain $\mathfrak{S}_a \subset \mathfrak{M} \times \mathfrak{M}$. The family of all $P(i, \tilde{\mathfrak{M}}, \varepsilon)$ for arbitrary bounded sets $\tilde{\mathfrak{M}} \supset \mathfrak{M}_B$ and any values of the positive number ε is taken as a fundamental system of neighborhoods of the point $i \in \mathfrak{M}$. Note that the neighborhood $P(i, \tilde{\mathfrak{M}}, \varepsilon)$ is a bounded set, because $\tilde{\mathfrak{M}} \supset \mathfrak{M}_B$. The subsets of $\mathfrak{M}$ containing a neighborhood of i in the fundamental system can be regarded as arbitrary neighborhoods $P(i)$ of i.

LEMMA 1. *The system of sets $P(i)$ introduced for each point $i \in \mathfrak{M}$ satisfies the axioms for a neighborhood system and determines on $\mathfrak{M}$ a unique separated (in the Hausdorff sense) topological structure.*

In what follows, it is convenient to understand $P(i)$ to be an open neighborhood of a point $i \in \mathfrak{M}$. The topology in $\mathfrak{M} \times \mathfrak{M}$ is defined in the usual way as the product of the topologies in the factors. It is natural to assume that the domain of a is open in $\mathfrak{M} \times \mathfrak{M}$, i.e., a pair $\langle ij \rangle$ belongs to $\mathfrak{S}_a$ along with some neighborhood $P(i) \times P(j)$ of it.

LEMMA 2. *The function $a\colon \mathfrak{M} \times \mathfrak{M} \to R$ is continuous in the topology of the direct product $\mathfrak{M} \times \mathfrak{M}$.*

Observe that if $a\colon \mathfrak{M} \times \mathfrak{M} \to R$ satisfies the axioms of an ordinary metric, then the topology constructed in $\mathfrak{M}$ coincides with the natural metric space topology and is the weakest topology for which this metric is continuous.

Let $n \geqslant 1$ be an arbitrary integer. For some n-tuple $\langle p \cdots q \rangle$ we construct the continuous mapping $a[p \cdots q]\colon \mathfrak{M} \to R^n$ assigning to an point $i \in \mathfrak{M}$ the n-tuple of numbers $(a(ip), \ldots, a(iq)) \in R^n$. The second condition defines the dimension of $\mathfrak{M}$.

II. For each point $i \in \mathfrak{M}$ there is an n-tuple $\langle p \cdots q \rangle$ such that the mapping $a[p \cdots q]\colon P(i) \to R^n$ is a local homeomorphism for some neighborhood $P(i)$.

According to this condition, $\mathfrak{M}$ is an n-dimensional topological manifold such that local coordinates $x^1, \ldots, x^n$ can be introduced in some neighborhood of each point by setting, for example, $x^1(i) = a(ip), \ldots, x^n(i) = a(iq)$. In some neighborhood $P(i) \times P(j)$ of any pair $\langle ij \rangle \in \mathfrak{S}_a$ the original function $a\colon \mathfrak{M} \times \mathfrak{M} \to R$ has a local coordinate representation

$$(1) \qquad a(ij) = a(x^1(i)s, \ldots, x^n(i); x^1(j), \ldots, x^n(j)),$$

whose properties are specified by the third condition.

III. The function $a(ij) = a(x^1(i), \ldots, x^n(i); x^1(j), \ldots, x^n(j))$ is sufficiently smooth, and the local coordinates appear in it in an essential way.

Sufficient smoothness of the function $a(ij)$ is understood as the existence of continuous derivatives of sufficiently high order. The essential dependence on the local coordinates presumes that their number cannot be reduced by a nonsingular substitution.

Suppose next that $m = n + 2 \geqslant 3$ and let $\mathfrak{M}^m$ be the m-fold direct product of $\mathfrak{M}$ with itself, with m-tuples as elements. We construct a mapping $A\colon \mathfrak{M}^m \to R^{m(m-1)/2}$ assigning to an m-tuple $\langle ijk \cdots vw \rangle \in \mathfrak{M}^m$, where $m = n + 2$, the $(m(m-1)/2)$-tuple $(a(ij), a(ik), \ldots, a(vw))$ of numbers corresponding to all ordered pairs in the m-tuple and regarded as the coordinates of some point in the space $R^{m(m-1)/2}$. Let $\mathfrak{S}_A$ denote the domain of this mapping; $\mathfrak{S}_A$ is obviously open in $\mathfrak{M}^m$. It is natural to assume that $\mathfrak{S}_A$ is nonempty and that any pair in $\mathfrak{S}_a$ belongs to some m-tuple in $\mathfrak{S}_A$. Neighborhoods of an m-tuple $\langle ijk \cdots vw \rangle$ in $\mathfrak{M}^m$ will be denoted by $P(\langle ijk \cdots vw \rangle)$.

LEMMA 3. *The mapping* $A\colon \mathfrak{M}^m \to R^{m(m-1)/2}$ *is continuous in the topology of the direct product* $\mathfrak{M}^m$.

DEFINITION 2. A function $a\colon \mathfrak{M} \times \mathfrak{M} \to R$ is said to give a *phenomenologically symmetric n-dimensional distance geometry of rank* $m = n + 2$ *on the set* $\mathfrak{M}$ if the following *axiom* is satisfied in addition to conditions I, II and III:

IV. Each m-tuple $\langle ijk \cdots vw \rangle$ $(m = n + 2)$ in a dense subset of $\mathfrak{S}_A \subset \mathfrak{M}^m$ has a neighborhood $P(\langle ijk \cdots vw \rangle)$ for which there exists a sufficiently smooth function Φ: $\mathfrak{E} \to R$ defined in some domain $\mathfrak{E} \subset R^{m(m-1)/2}$ such that $\operatorname{grad} \Phi \neq 0$ at the point $A(\langle ijk \cdots vw \rangle) \in \mathfrak{E}$ and the set $A(p(\langle ijk \cdots vw \rangle))$ coincides with the set of zeros of Φ, i.e.,

$$\Phi(a(ij), a(ik), \ldots, a(vw)) = 0 \tag{2}$$

for every m-tuple in $P(\langle ijk \cdots vw \rangle)$.

Axiom IV amounts to the principle of phenomenological symmetry in the general scheme of the theory of physical structures proposed by Kulakov [3] as a means of classifying physical laws. This axiom expresses the requirement that the $m(m-1)/2$ ordered mutual distances between the points of any m-tuple in $P(\langle ijk \cdots vw \rangle)$ are related in a nontrivial way, i.e., they satisfy some equation (2) giving an analytic expression for a physical law. The requirement in axiom IV that $\operatorname{grad} \Phi \neq 0$ at the point $A(\langle ijks \cdots vw \rangle)$ means, roughly speaking, that $\mathfrak{M}^m$ contains an m-tuple $\langle ijk \cdots vw \rangle$ in general position, i.e., the mapping $A\colon \mathfrak{M}^m \to R^{m(m-1)/2}$ is nonsingular in a definite sense.

LEMMA 4. *The set of m-tuples* $\langle ijk \cdots vw \rangle$ *for which all the first-order derivatives of* Φ *at the corresponding point* $A(\langle ijk \cdots vw \rangle)$ *are nonzero is dense in* $\mathfrak{S}_A \subset \mathfrak{M}^m$.

Using the representation (1), we write the local coordinate specification of the mapping $A\colon \mathfrak{M}^m \to R^{m(m-1)/2}$:

$$\begin{aligned} a(ij) &= a(x^1(i), \ldots, x^n(i); x^1(j), \ldots, x^n(j)), \\ a(ik) &= a(x^1(i), \ldots, x^n(i); x^1(k), \ldots, x^n(k)), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a(vw) &= a(x^1(v), \ldots, x^n(v); x^1(w), \ldots, x^n(w)). \end{aligned} \tag{3}$$

The specification (3) of the mapping A is a specification of $m(m-1)/2$ differentiable functions $a(ij), a(ik), \ldots, a(vw)$ depending in a special and essential way on the mn local coordinates $x^1(i), \ldots, x^n(i), \ldots, x^n(w)$. Since $m = n + 2 \geqslant 3$, we have $m(m-1)/2 \leqslant mn$, i.e., the number of functions is less than the number of coordinates and, therefore, the connection (2) is a nontrivial fact. The Jacobi matrix of the system (3) is the functional matrix of A, and its rank is called the *rank* of this mapping.

THEOREM 1. *A function* a: $\mathfrak{M} \times \mathfrak{M} \to R$ *gives a phenomenologically symmetric n-dimensional distance geometry of rank* $m = n + 2$ *on the set* $\mathfrak{M}$ *if and only if the rank of the mapping* A: $\mathfrak{M}^m \to R^{m(m-1)/2}$ *is equal to* $m(m-1)/2 - 1$ *on a dense subset of* $\mathfrak{S}_A \subset \mathfrak{M}^m$.

As usual, a one-to-one mapping of $\mathfrak{M}$ onto itself is called a *transformation* of $\mathfrak{M}$. A transformation is said to be a *motion* (an *isometry*) if it preserves the original function (metric) a: $\mathfrak{M} \times \mathfrak{M} \to R$. The collection of all transformations with respect to which the metric a is a two-point invariant is obviously a group. The coordinate representation of a differentiable local transformation of $\mathfrak{M}$ can be written in the form

$$(4) \qquad x'^{\mu} = \lambda^{\mu}(x^1, x^2, \dots, x^n), \qquad \mu = 1, 2, \dots, n.$$

The invariance of the metric (1) with respect to the transformations (4) means that

$$(5) \quad a(x^1(i), \dots, x^n(i); x^1(j), \dots, x^n(j)) = a(\lambda^1(i), \dots, \lambda^n(i); \lambda^1(j), \dots, \lambda^n(j)),$$

where, for example, $\lambda^{\nu}(i) = \lambda^{\mu}(x^1(i), \dots, x^n(i))$. From the known function (1) we can find the group of transformations (4) by solving (5). However, we know only that the metric (1) is phenomenologically invariant, i.e., it satisfies some equation (2). But this turns out to be sufficient for establishing the existence of an $(n(n+1)/2)$-parameter group of motions which gives a group symmetry of the n-dimensional space $\mathfrak{M}$.

DEFINITION 3. We say that a function a: $\mathfrak{M} \times \mathfrak{M} \to R$ gives an *n-dimensional distance geometry on* $\mathfrak{M}$ *provided with a group symmetry of degree* $n(n+1)/2$ if the following axiom is satisfied in addition to conditions I, II and III:

IV′. For any pair $\langle ij \rangle$ in a dense subset of $\mathfrak{S}_a \subset \mathfrak{M} \times \mathfrak{M}$ there exists a local group of differentiable local transformations (motions) of some neighborhood $P(i) \times P(j)$ of $\langle ij \rangle$ containing at most $n(n+1)/2$ essential independent parameters such that the function a: $P(i) \times P(j) \to R$ is a two-point invariant.

The local group of transformations referred to in axiom IV′ determines the complete mobility of rigid bodies in $\mathfrak{M}$ with $n(n+1)/2$ degrees of freedom. However, in the general case the motion is not given for all the points in $\mathfrak{M}$, just as the function a is not defined for all the pairs in $\mathfrak{M} \times \mathfrak{M}$. It is also possible that the distance $a(ij)$ is not defined, while for certain neighborhoods $P(i)$ and $P(j)$ the transformation (4) is given.

LEMMA 5. *The set of m-tuples* $\langle ijk \cdots vw \rangle$ $(m = n + 2)$ *having a neighborhood* $P(i) \times P(j) \times \cdots \times P(w)$ *whose motion preserves all the ordered distances* a: $P(i) \times P(j) \to R$, a: $P(i) \times P(k) \to R, \dots,$ a: $P(v) \times P(w) \to R$ *is dense in* $\mathfrak{S}_A \subset \mathfrak{M}^m$.

THEOREM 2. *A function* a: $\mathfrak{M} \times \mathfrak{M} \to R$ *gives an n-dimensional distance geometry on* $\mathfrak{M}$ *equipped with a group symmetry of degree* $n(n+1)/2$ *if and only if the rank of the mapping* A: $\mathfrak{M}^m \to R^{m(m-1)/2}$ *is equal to* $m(m-1)/2 - 1$ *on a dense subset of* $\mathfrak{S}_A \subset \mathfrak{A}^m$.

The summary result of this note is the establishment of equivalence for the phenomenological and group symmetries of an n-dimensional distance geometry. This equivalence follows directly from Theorems 1 and 2.

THEOREM 3. *A function* a: $\mathfrak{M} \times \mathfrak{M} \to R$ *gives a phenomenologically symmetric n-dimensional distance geometry of rank* $m = n + 2$ *on* $\mathfrak{M}$ *if and only if this function gives an n-dimensional distance geometry equipped with a group symmetry of degree* $n(n+1)/2$ *on* $\mathfrak{M}$.

In conclusion we mention that the necessary and sufficient condition in Theorems 1 and 2 on the rank of A could be included in the definition of an n-dimensional distance geometry which, on the one hand, would be phenomenologically symmetric and, on the other hand, would be equipped with a group symmetry, and both symmetries would be completely equivalent, according to Theorem 3.

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