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PHENOMENOLOGICAL AND GROUP SYMMETRY IN THE GEOMETRY OF TWO SETS (THEORY OF PHYSICAL STRUCTURES)

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In studying the foundations of physics, Kulakov [1] proposed a mathematical model of the structure of a physical law as a phenomenologically invariant connection between quantities measurable in experiment. This model, called a physical structure, has geometric character, is applicable to ordinary geometry [2], [3], and can be considered as a particular geometry of two sets. In the new geometry it is possible to introduce motion as a transformation which preserves the distance between points of different sets and defines its group symmetry. In this note the phenomenological and group symmetries of the geometry of two sets are defined precisely, and their complete equivalence is established.

Suppose there are two sets $\mathfrak{M}$ and $\mathfrak{N}$ of arbitrary (in general, different) nature whose points we denote by lower case Latin and Greek letters respectively, and suppose there is a function $a\colon \mathfrak{M} \times \mathfrak{N} \to R$ which assigns to the pair $\langle i\alpha\rangle \in \mathfrak{M} \times \mathfrak{N}$ a real number $a(i\alpha) \in R$. We note that the domain of the function a, which we denote by $\mathfrak{S}_a$, may not coincide with the entire direct product $\mathfrak{M} \times \mathfrak{N}$, i.e., a number is not assigned to each pair $\langle i\alpha\rangle \in \mathfrak{M} \times \mathfrak{N}$. Below, however, it is convenient not to stipulate this circumstance each time, assuming that a pair $\langle i\alpha\rangle$ is always taken from the domain $\mathfrak{S}_a$. The function $a\colon \mathfrak{M} \times \mathfrak{N} \to R$ can be considered as a kind of metric in the geometry of two sets. The distance $a(i\alpha)$ is defined for points $i \in \mathfrak{M}$ and $\alpha \in \mathfrak{M}$ of different sets, and hence the usual properties of a metric (such as symmetry, the triangle inequality, etc.) do not hold here. We shall assume that the following condition is satisfied:

I. There exists a finite basis set $\mathfrak{N}_B$ (respectively, $\mathfrak{M}_B$) such that if two arbitrary points $i, j \in \mathfrak{M}$ (respectively, $\alpha, \beta \in \mathfrak{N}$) are distinct, then for some $\gamma \in \mathfrak{N}_B$ (respectively, $k \in \mathfrak{M}_B$) the inequality $a(i\gamma) \neq a(j\gamma)$ holds (respectively, $a(k\alpha) \neq a(k\beta)$).

The meaning of condition I is primarily that only those properties of the sets $\mathfrak{M}$ and $\mathfrak{N}$ are considered which can be expressed by means of the function a. If the set $\mathfrak{M}$ (respectively, $\mathfrak{N}$) contains more than two distinct points, then by condition I for any point $i \in \mathfrak{M}$ (respectively, $\alpha \in \mathfrak{N}$) there is a $\gamma \in \mathfrak{N}_B$ (respectively, $k \in \mathfrak{M}_B$) such that the pair $\langle i\gamma\rangle$ (respectively, $\langle k\alpha\rangle$) belongs to the domain of the function a, i.e., $\mathrm{pr}_1 \mathfrak{S}_a = \mathfrak{M}$ (respectively, $\mathrm{pr}_2 \mathfrak{S}_a = \mathfrak{N}$).

DEFINITION 1. A set $\tilde{\mathfrak{M}} \subset \mathfrak{M}$ (respectively, $\tilde{\mathfrak{N}} \subset \mathfrak{N}$) is called *bounded* if the range of the function $a\colon \tilde{\mathfrak{M}} \times \mathfrak{N}_B \to R$ (respectively, $a\colon \mathfrak{M}_B \times \tilde{\mathfrak{N}} \to R$) is bounded.

It is clear that any finite set is bounded, and the union of finitely many bounded sets is bounded.

It is possible to define topologies in the sets $\mathfrak{M}$ and $\mathfrak{N}$ in a natural way. Suppose, for example, $i \in \mathfrak{M}$ is an arbitrary point, $\tilde{\mathfrak{N}}$ is a bounded set containing the basis $\mathfrak{N}_B$, and $\varepsilon > 0$. We denote by $P(i, \tilde{\mathfrak{N}}, \varepsilon)$ the set of all points $i' \in \mathfrak{M}$ for which the inequality $|a(i'\gamma) - a(i\gamma)| < \varepsilon$ holds for any $\gamma \in \tilde{\mathfrak{N}}$ such that the pair $\langle i\gamma\rangle$ belongs to $\mathfrak{S}_a$. The family of all $P(i, \tilde{\mathfrak{N}}, \varepsilon)$ for arbitrary bounded sets $\tilde{\mathfrak{N}} \supset \mathfrak{N}_B$ and any values of the positive

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number ε is taken as a fundamental system of neighborhoods of the point $i \in \mathfrak{M}$. For any point $\alpha \in \mathfrak{N}$ a fundamental system of neighborhoods $P(\alpha, \tilde{\mathfrak{M}}, \varepsilon)$ is introduced in a similar way. We denote arbitrary neighborhoods of points i and α by $P(i)$ and $P(\alpha)$.

LEMMA 1. *The system of sets $P(i)$ and $P(\alpha)$ introduced for all points $i \in \mathfrak{M}$ and $\alpha \in \mathfrak{N}$ satisfy the axioms of a system of neighborhoods and define on the sets $\mathfrak{M}$ and $\mathfrak{N}$ unique topological structures which are separable in the sense of Hausdorff.*

The topology in $\mathfrak{M} \times \mathfrak{N}$ is defined in the usual way as the product of the topologies of the factor spaces. It is natural to suppose that the domain of the function a is open in $\mathfrak{M} \times \mathfrak{N}$, i.e., a pair $\langle i\alpha \rangle$ belongs to $\mathfrak{S}_a$ together with some neighborhood $P(i) \times P(\alpha)$ of it, and, moreover, $\mathfrak{S}_a \supset \mathfrak{M}_B \times \mathfrak{N}_B$, since bases of the sets $\mathfrak{M}_B$ and $\mathfrak{N}_B$ are finite by condition I.

LEMMA 2. *The function $a \colon \mathfrak{M} \times \mathfrak{N} \to R$ is continuous in the topology of the direct product $\mathfrak{M} \times \mathfrak{N}$.*

Suppose $m \geq 1$ (respectively, $n \geq 1$) is an arbitrary integer. For some cortege $\langle \delta \cdots \rho \rangle$ of length m (respectively, $\langle p \cdots q \rangle$ of length n) we construct a continuous mapping $a[\delta \cdots \rho] \colon \mathfrak{M} \to R^m$ (respectively, $a[p \cdots q] \colon \mathfrak{N} \to R^n$) by assigning to a point $i \in \mathfrak{M}$ (respectively, $\alpha \in \mathfrak{N}$) the set of m numbers $(a(i\delta), \ldots, a(i\rho)) \in R^m$ (respectively, the n numbers $(a(p\alpha), \ldots, a(q\alpha)) \in R^n$). The second condition defines the dimension of the sets $\mathfrak{M}$ and $\mathfrak{N}$.

II. For each point $i \in \mathfrak{M}$ (respectively, $\alpha \in \mathfrak{N}$) there is a cortege $\langle \delta \cdots \rho \rangle \in \mathfrak{N}_B^m$ of length m (respectively, $\langle p \cdots q \rangle \in \mathfrak{M}_B^n$ of length n) such that for some neighborhood $P(i)$ (respectively, $P(\alpha)$) the mapping $a[\delta \cdots \rho] \colon P(i) \to R^m$ (respectively, $a[p \cdots q] \colon P(\alpha) \to R^n$) is a local homeomorphism.

According to condition II the sets $\mathfrak{M}$ and $\mathfrak{N}$ are topological manifolds of dimension m and n in some neighborhood of each point of which it is possible to introduce local coordinates $x^1, \ldots, x^m$ and $\xi^1, \ldots, \xi^n$. For the original function $a \colon \mathfrak{M} \times \mathfrak{N} \to R$ in some neighborhood $P(i) \times P(\alpha)$ of an arbitrary pair $\langle i\alpha \rangle \in \mathfrak{S}_a$ we obtain the local coordinate representation

$$a(i\alpha) = \Phi a(x^1(i), \ldots, x^m(i), \xi^1(\alpha), \ldots, \xi^n(\alpha)), \tag{1}$$

whose properties are given by the third condition.

III. The function $a(i\alpha)$ is sufficiently smooth, and the local coordinates $x^1(i), \ldots, x^m(i)$ and $\xi^1(\alpha), \ldots, \xi^n(\alpha)$ are contained in it in an essential manner.

Essential dependence on the coordinates means that no nondegenerate change leads to a reduction of the number of them.

Suppose, further, that $\mathfrak{M}^{n+1}$ and $\mathfrak{N}^{m+1}$ are the $(n+1)$-fold and $(m+1)$-fold direct products of the sets $\mathfrak{M}$ and $\mathfrak{N}$ with themselves. In $\mathfrak{M}^{n+1}$, $\mathfrak{N}^{m+1}$ and also $\mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$ we define the direct-product topologies in the usual manner. We construct a mapping $A \colon \mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1} \to R^{(n+1)(m+1)}$ by assigning to a cortege

$$\langle ijk \cdots v, \alpha\beta\gamma \cdots \tau \rangle \in \mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$$

of length $n+m+2$ the set of $(n+1)(m+1)$ numbers $(a(i\alpha), a(i\beta), \ldots, a(v\tau))$ corresponding to all ordered pairs in the cortege. We denote the domain of the mapping thus constructed by $\mathfrak{S}_A$. A neighborhood of the cortege $\langle ijk \cdots v, \alpha\beta\gamma \cdots \tau \rangle$ in $\mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$ we denote by $P(\langle ijk \cdots v, \alpha\beta\gamma \cdots \tau \rangle)$. Since the domain $\mathfrak{S}_a$ by hypothesis is open in $\mathfrak{M} \times \mathfrak{N}$, the domain $\mathfrak{S}_A$ in $\mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$ is also open.

LEMMA 3. *The mapping $A \colon \mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1} \to R^{(n+1)(m+1)}$ is continuous in the direct product topology of $\mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$.*

DEFINITION 2. We shall say that *the function* $a\colon \mathfrak{M} \times \mathfrak{N} \to R$ *defines on the sets* $\mathfrak{M}$ *and* $\mathfrak{N}$ *a phenomenologically symmetric* $(m+n)$*-dimensional geometry of distances of rank* $(n+1, m+1)$ if, in addition to conditions I, II, and III, the following axiom holds.

IV. For each cortege $\langle ijk\cdots v, \alpha\beta\gamma\cdots\tau\rangle$ of length $m+n+2$ from a set dense in $\mathfrak{S}_A \subset \mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$ and some neighborhood $P(\langle ijk\cdots v, \alpha\beta\gamma\cdots\tau\rangle)$ of it there exists a sufficiently smooth function $\Phi\colon \mathcal{E} \to R$ defined in some region $\mathcal{E} \subset R^{(n+1)(m+1)}$ such that $\operatorname{grad}\Phi \neq 0$ at the point $A(\langle ijk\cdots v, \alpha\beta\gamma\cdots\tau\rangle) \in \mathcal{E}$ and the set $A(P(\langle ijk\cdots v, \alpha\beta\gamma\cdots\tau\rangle))$ is a subset of the set of zeros of the function Φ, i.e.,

$$\Phi(a(i\alpha), a(i\beta), \ldots, a(v\tau)) = 0 \tag{2}$$

for each cortege in $P(\langle ijk\cdots v, \alpha\beta\gamma\cdots\tau\rangle)$.

Axiom IV comprises the content of the principle of phenomenological symmetry. The requirement $\operatorname{grad}\Phi \neq 0$ at the point $A(\langle ijk\cdots v, \alpha\beta\gamma\cdots\tau\rangle)$ means that the mapping A is nondegenerate in a certain sense.

The metrics of all existing phenomenologically symmetric geometries of two sets of any dimension and also equations (2) for each such metric were presented in the author's note [4].

Using the representation (1), we can write out a local coordinate definition of the mapping A which is the definition of a system of $(n+1)(m+1)$ differentiable functions $a(i\alpha), a(i\beta), \ldots, a(v\tau)$ depending in a special way on the $m(n+1)+n(m+1)$ coordinates $x^1(i), \ldots, x^m(i), \ldots, \xi^1(\tau), \ldots, \xi^n(\tau)$. The Jacobi matrix of this system of functions is the functional matrix of the mapping A, and its rank is called the rank of this mapping.

THEOREM 1. *The function* $a\colon \mathfrak{M} \times \mathfrak{N} \to R$ *defines on the sets* $\mathfrak{M}$ *and* $\mathfrak{N}$ *a phenomenological symmetric* $(m+n)$*-dimensional geometry of distances of rank* $(n+1, m+1)$ *if and only if the rank of the mapping* A *is equal to* $(n+1)(m+1) - 1$ *on a dense set in* $\mathfrak{S}_A \subset \mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$.

As we know, a transformation of the sets $\mathfrak{M}$ and $\mathfrak{N}$ is defined to be a one-to-one mapping of these sets onto themselves. We call a transformation a *motion* if it preserves the original function (the metric) $a\colon \mathfrak{M} \times \mathfrak{N} \to R$. The set of all transformations under which the metric a is a two-point invariant (i.e. is preserved) is obviously a group. For example, for the metric $a(i\alpha) = x(i)\xi(\alpha) + \eta(\alpha)$ taken from [4] a two-parameter group of motions is given by the equations $x' = bx + c$, $\xi' = \xi/b$ and $\eta' = \eta - c\xi/b$. Regarding the metric (1), we know only that it is phenomenologically invariant, i.e., satisfies some equation (2). But this turns out to be sufficient to establish the existence of a group of motions with mn parameters.

DEFINITION 3. We say that *the function* $a\colon \mathfrak{M} \times \mathfrak{N} \to R$ *defines on the sets* $\mathfrak{M}$ *and* $\mathfrak{N}$ *an* $(m+n)$*-dimensional geometry of distances equipped with a group symmetry of degree* mn if, in addition to conditions I, II, and III, the following axiom holds.

IV$'$. For each point i of a denset set in $\mathfrak{M}$ and for each point α of a sense set in $\mathfrak{N}$ there exists a local group of sufficiently smooth local transformations of some neighborhoods $P(i)$ and $P(\alpha)$ of them containing mn essential independent parameters such that if the pair $\langle i\alpha\rangle \in \mathfrak{S}_a$, then the function $a\colon P(i) \times P(\alpha) \to R$ is a two-point invariant.

The local group of transformations mentioned in axiom IV$'$ determines the total mobility of "solid bodies" in the $(m+n)$-dimensional space $\mathfrak{M} \times \mathfrak{N}$ with mn degrees of freedom. However, in the general case such a motion is not defined for every pair in $\mathfrak{S}_a$.

LEMMA 4. *The set of corteges* $\langle ijk\cdots v, \alpha\beta\gamma\cdots\tau\rangle$ *of length* $n+m+2$*, the motion of some neighborhood* $P(i) \times \cdots \times P(\tau)$ *of which containing* mn *essential independent parameters preserves all distances* $a\colon P(i) \times P(\alpha) \to R$, $a\colon P(i) \times P(\beta) \to R, \ldots, a\colon P(v) \times P(\tau) \to R$, *is dense in* $\mathfrak{S}_A \subset \mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$.

THEOREM 2. *The function* $a\colon \mathfrak{M} \times \mathfrak{N} \to R$ *defines on the sets* $\mathfrak{M}$ *and* $\mathfrak{N}$ *an* $(m+n)$*-dimensional geometry of distances equipped with a group symmetry of degree* mn *if and* *only if the rank of the mapping A is equal to $(n+1)(m+1)-1$ on a set dense in $\mathfrak{S}_A \subset \mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$.*

Our final result consists in establishing the equivalence of phenomenological and group symmetry of an $(m+n)$-dimensional geometry of the distances of two sets. This equivalence follows immediately from the two theorems formulated above.

THEOREM 3. *The function $a\colon \mathfrak{M} \times \mathfrak{N} \to R$ defines on the sets $\mathfrak{M}$ and $\mathfrak{N}$ a phenomenologically symmetric $(m+n)$-dimensional geometry of distances of rank $(n+1, m+1)$ if and only if this function defines on $\mathfrak{M}$ and $\mathfrak{N}$ an $(m+n)$-dimensional geometry of distances equipped with a group symmetry of degree mn.*

We note that since all phenomenologically invariant metrics of an $(m+n)$-dimensional geometry of distances of two sets are known [4], Theorem 3 provides a solution of the following problem of the theory of invariants of transformation groups. Suppose that on the direct product $R^m \times R^n$ there acts mn-dimensional group of transformations $x' = \lambda(x, \varphi)$, $\xi' = \sigma(\xi, \varphi)$, where $x, x' \in R^m$, $\xi, \xi' \in R^n$ and $\varphi \in R^{mn}$, the explicit form of these transformations being unknown. A function $a(x, \xi)$ of a pair of points x and ξ will be a two-point invariant if it is preserved under transformation, i.e., satisfies the following equation:

$$a(x', \xi') = a(x, \xi).$$

It is required, first, to establish for what pairs of numbers m, n there exist locally nondegenerate, two-point invariants, second, to find explicit expressions for these invariants, and third, to write down an mn-dimensional group of transformations of the space $R^m \times R^n$ which preserves the function $a(x, \xi)$. The solution of this problem may turn out to be complicated for the reason that there is no complete classification of the transformation groups.

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